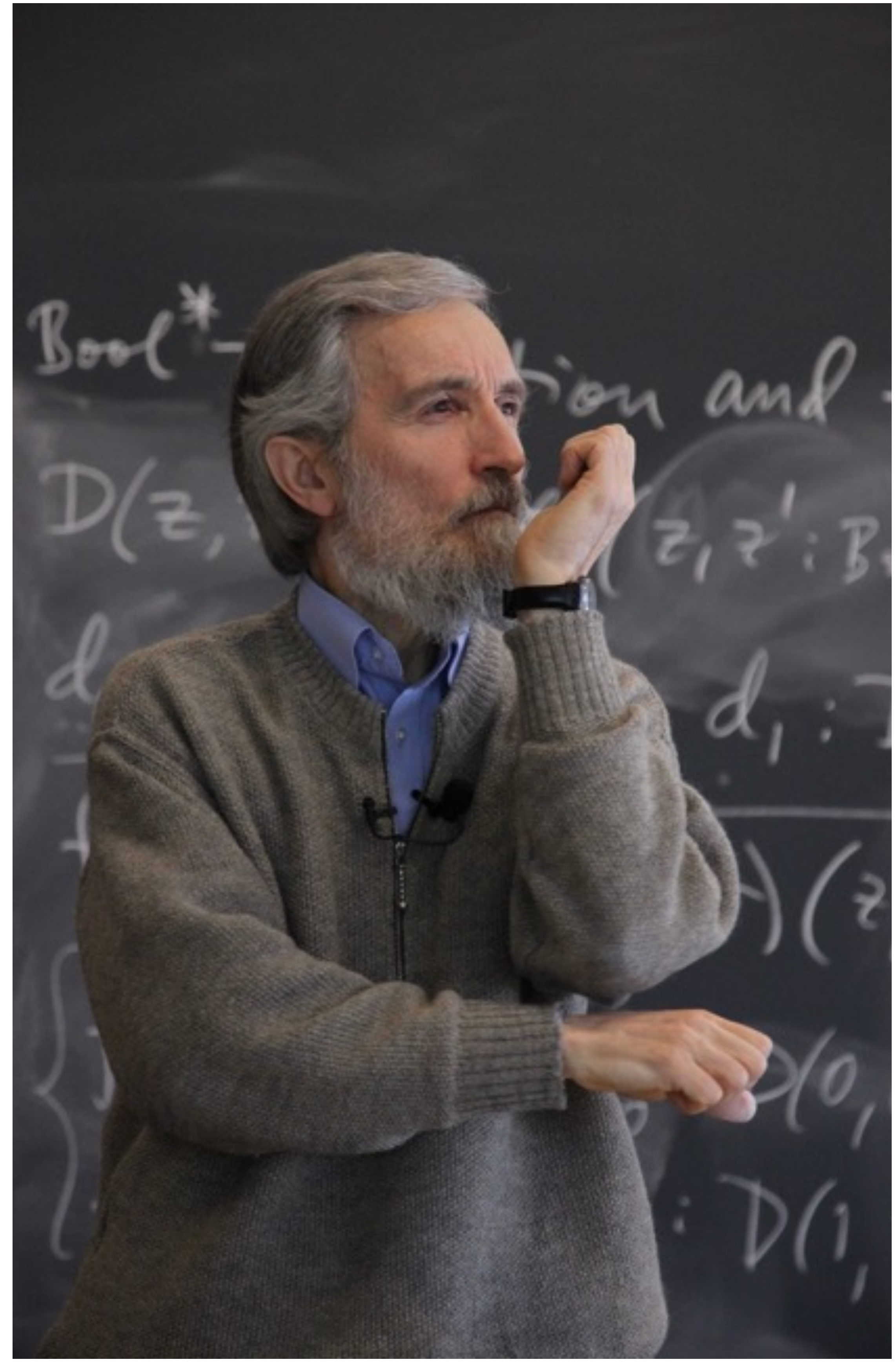
# Auto in Agda

joint work with Pepijn Kokke

APLS
Frankfurt, December 2015

# Per Martin-Löf

*“The intuitionistic type theory,…, may equally well be viewed as a programming language.”* – Constructive Mathematics and Computer Programming ‘79



Type theory provides
a *single language*
for proofs, programs, and specs.

# Coq

# Coq

- Gallina – a small functional programming language

# Coq

- Gallina – a small functional programming language
- Tactics – commands that generate proof terms

# Coq

- Gallina – a small functional programming language
- Tactics – commands that generate proof terms
- Ltac – a tactic scripting language

# Coq

- Gallina – a small functional programming language
- Tactics – commands that generate proof terms
- Ltac – a tactic scripting language
- ML-plugins – add custom tactics to proof assistant

# Coq

- Gallina – a small functional programming language
- Tactics – commands that generate proof terms
- Ltac – a tactic scripting language
- ML-plugins – add custom tactics to proof assistant

**What happened to the idea of a single language?**

# Introducing Agda

```
data Even  : ℕ →  Set where
   Base : Even 0
   Step : Even n → Even (suc (suc n))
```

# Introducing Agda

```
data Even  : ℕ →  Set where
   Base : Even 0
   Step : Even n → Even (suc (suc n))

even4 : Even 4
even4 = Step (Step Base)
```

# Introducing Agda

```
data Even  : ℕ →  Set where
    Base : Even 0
    Step : Even n → Even (suc (suc n))

even4 : Even 4
even4 = Step (Step Base)

even1024 : Even 1024
even1024 = …
```

# A definition that computes

```
data Empty : Set where

data True : Set where
  tt : True

even? : ℕ -> Set
even? zero          = True
even? (suc zero)    = Empty
even? (suc (suc n)) = even? n
```

# A definition that computes

```
data Empty : Set where

data True : Set where
  tt : True

even? : ℕ -> Set
even? zero            = True
even? (suc zero)      = Empty
even? (suc (suc n))   = even? n

even1024 : even? 1024
even1024 = tt
```

# Proof-by-reflection

```
soundness : (n : ℕ) -> even? n -> Even n
soundness zero          e   = Base
soundness (suc zero)    ()
soundness (suc (suc n)) e   = Step (soundness n e)

even1024 : Even 1024
even1024 = soundness 1024 tt
```

# Proof-by-reflection

- Works very well for *closed problems,* without variables or additional hypotheses

- You can implement ‘solvers’ for a fixed domain (such as Agda’s monoid solver or ring solver), although there may be some ‘syntactic overhead’.

- But sometimes the automation you would like is more *ad-hoc*.

# Even – again

```
even+ : Even n -> Even m -> Even (n + m)
even+ Base       e2 = e2
even+ (Step e1) e2 = Step (even+ e1 e2)

simple : ∀ n → Even n → Even (n + 2)
simple = …
```

# Even – again

```
even+ : Even n -> Even m -> Even (n + m)
even+ Base        e2 = e2
even+ (Step e1) e2 = Step (even+ e1 e2)

simple : ∀ n → Even n → Even (n + 2)
simple = …
```

We need to give a proof term by hand…

# Maintaining hand-written proofs

- **Brittle**
- **Large**
- **Incomplete**

# Even – automatic

```
even+ : Even n -> Even m -> Even (n + m)
even+ Base       e2 = e2
even+ (Step e1) e2 = Step (even+ e1 e2)

simple : ∀ {n} → Even n → Even (n + 2)
simple = tactic (auto 5 hints)
```

The auto function performs proof search,
trying to prove the current goal from some list of ‘hints’

# Even – again

```
even+ : Even n -> Even m -> Even (n + m)
even+ Base      e2 = e2
even+ (Step e1) e2 = Step (even+ e1 e2)

simple : ∀ {n} → Even n → Even (4 + n)
simple = tactic (auto 5 hints)
```

Our definition is now more robust.
Reformulating the lemma does not need proof refactoring.

Use reflection
to *generate* proof terms

# Agda's reflection mechanism

- A built-in type `Term`
- Quoting a term, `quoteTerm`, or goal, `quoteGoal`
- Unquoting a value of type term, splicing back the corresponding concrete syntax.

```
data Term : Set where
  -- Variable applied to arguments.
  var     : (x : ℕ) (args : List (Arg Term)) → Term
  -- Constructor applied to arguments.
  con     : (c : Name) (args : List (Arg Term)) → Term
  -- Identifier applied to arguments.
  def     : (f : Name) (args : List (Arg Term)) → Term
  -- Different flavours of λ-abstraction.
  lam     : (v : Visibility) (t : Term) → Term
  -- Pi-type.
  pi      : (t₁ : Arg Type) (t₂ : Type) → Term
  …
```

# Automation using reflection

```
even+ : Even n -> Even m -> Even (n + m)
even+ Base        e2 = e2
even+ (Step e1) e2 = Step (even+ e1 e2)

simple : ∀ {n} → Even n → Even (n + 2)
simple = quoteGoal g in unquote(…g…)
```

# Automation using reflection

```
even+ : Even n -> Even m -> Even (n + m)
even+ Base        e2 = e2
even+ (Step e1) e2 = Step (even+ e1 e2)

simple : ∀ {n} → Even n → Even (n + 2)
simple = tactic(λ g → …g…)
```

All I need to provide here is a function from `Term` to `Term`

# Examples

```
hints : HintDB
hints = [] << quote Base
           << quote Step
           << quote even+

test1 : Even 4
test1 = tactic (auto 5 hints)

test2 : ∀ {n} → Even n → Even (n + 2)
test2 = tactic (auto 5 hints)

test3 : ∀ {n} → Even n → Even (4 + n)
test3 = tactic (auto 5 hints)
```

# How auto works

1. Quote the current goal;
2. Translate the goal to our own Term data type;
3. First-order proof search with this Term as goal;
4. Build an Agda Term from the result;
5. Unquote this final Term.

# Proof automation in Agda

1. Quote the current goal;
2. Translate the goal to our own Term data type;
3. **First-order proof search with this Term as goal;**
4. Build an Agda AST from this result;
5. Unquote the AST.

# Prolog-style resolution

while there are open goals

  try to apply each rule to resolve the next goal

  if this succeeds

    add premises of the rule to the open goals

    continue the resolution

  otherwise fail and backtrack

# Implementing auto

- First convert the goal to our own (first-order) term type;
- if this fails, generate an error term;
- otherwise, build up a search tree and traverse some fragment of this tree.
- if this produces at least one proof, turn it into a built-in term, ready to be unquoted.
- if this doesn’t find a solution, generate an error term.

# Handling failure

- In this way we are searching a *infinite* search space
- Yet all Agda programs must be total and terminate.
- We *coinductively* construct a search tree;
- The user may traverse a finite part of this tree in search of a solution..

# Finding solutions

- We can use a simple depth-bounded search

```
dbs : (depth : ℕ) → SearchTree A → A
```

- Or implement breadth-first search;
- Or any other traversal of the search tree.

# Alternatives

- Apply every rule at most once;
- Assign priorities to the rules;
- Limit when or how some rules are used.
- …

# Example - sublists

```
data Sublist : List a -> List a -> Set where
  Base : ∀ ys -> Sublist [] ys
  Keep : ∀ x xs ys -> Sublist xs ys -> Sublist (x :: xs) (x :: ys)
  Drop : ∀ x xs ys -> Sublist xs ys -> Sublist xs (x :: ys)

reflexivity : ∀ xs -> Sublist xs xs

transitivity : ∀ xs ys zs ->
  Sublist xs ys -> Sublist ys zs -> Sublist xs ys

sublistHints : HintDB
```

# Example – sublists

```
wrong : ∀ x -> Sublist (x :: []) []
wrong = tactic (auto 5 sublistHints)
```

What happens?

# Missing from the presentation

- Conversion from Agda’s Term to our Term type;
- Building an Agda Term to unquote from a list of rules that have been applied;
- Generating rules from lemma names.
- Implementation of unification and resolution.

# Discussion

- Lots of limitations:
  - first-order;
  - limited information from local context;
  - not very fast – and it's hard to tell how to fix this!

# Discussion

- Lots of limitations:
  - first-order;
  - limited information from local context;
  - not very fast – and it's hard to tell how to fix this!
- Constructing mathematics is indistinguishable from computer programming.