

Typing Coroutines

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Abstract. A coroutine is a programming construct between function and thread. It behaves like a function that can suspend itself arbitrarily often to yield intermediate results and to get new inputs before returning a result. This facility makes coroutines suitable for implementing generator abstractions.

Languages that support coroutines are often untyped or they use trivial types for coroutines. This work supplies the first type system with dedicated support for coroutines. The type system is based on the simply-typed lambda calculus extended with effects that describe control transfers between coroutines.

1 Introduction

A coroutine is a programming construct between function and thread. It can be invoked like a function, but before it returns a value (if ever) it may suspend itself arbitrarily often to return intermediate results and then be resumed with new inputs. Unlike with preemptive threading, a coroutine does not run concurrently with the rest of the program, but rather takes control until it voluntarily suspends to either return control to its caller or to pass control to another coroutine. Coroutines are closely related to cooperative threading, but they add value because they are capable of passing values into and out of the coroutine and they permit explicit switching of control.

Coroutines were invented in the 1960s as a means for structuring a compiler [4]. They have received a lot of attention in the programming community and have been integrated into a number of programming languages, for instance in Simula 67 [5], BETA, CLU [11], Modula-2 [19], Python [17], and Lua [15], and Knuth finds them convenient in the description of algorithms [8]. Coroutines are also straightforward to implement in languages that offer first-class continuations (e.g., Scheme [7]) or direct manipulation of the execution stack (e.g., assembly language, Smalltalk).

The main uses of coroutines are the implementation of compositions of state machines as in Conway’s seminal paper [4] and the implementation of generators. A generator enumerates a potentially infinite set of values with successive invocations. The latter use has led to renewed interest in coroutines and to their inclusion in mainstream languages like C# [13], albeit in restricted form as generators.

Despite the renewed interest in the programming construct per se, the typing aspects of coroutines have not received much attention. Indeed, the supporting languages are either untyped (e.g., Lua, Scheme, Python), the typing for coroutines is trivialized, or coroutines are restricted so that a very simple typing is sufficient. For instance, in Modula-2, coroutines are created from parameterless procedures so that all communication between coroutines must take place through global variables. Also, for describing generators, a simple function type seems sufficient.

Contribution. We propose a static type system for first-class, stackful coroutines that may be used in both, symmetric and asymmetric ways.¹ Moreover, we permit passing arguments to a coroutine at each start and resume operation, and we permit returning results on each suspend and on termination of the coroutine (and we distinguish between these two events). Our type system is based on the simply-typed lambda calculus. It includes an effect system that describes the way the coroutine operations are used. We present a small-step operational semantics for the language and prove type soundness.

Outline. Sec. 2 describes the language CorDuroy. It starts with some examples (Sec. 2.1) before delving into operational semantics (Sec. 2.2) and the type system (Sec. 2.3). Sec. 3 proves type soundness by establishing preservation and progress properties following the syntactic approach [20]. Sec. 4 discusses related work, and Sec. 5 concludes and outlines directions of further research.

2 CorDuroy

The language CorDuroy is a simply typed lambda calculus with recursive functions and operations for handling coroutines. Fig. 1 specifies the syntax; labels ℓ only occur at run time. We define λ -abstraction as sugar for the fixpoint operator: $\lambda x.e := \text{fix } \lambda_.\lambda x.e$.

Coroutines in CorDuroy are run-time entities identified by a label ℓ . The only way to create them is by applying the `create` operator to a function. Once a coroutine has been created, it can be executed. Unlike threads in a multi-threaded language, only one coroutine is active at any given time.

To activate a coroutine, there is a symmetric (`transfer`) and an asymmetric (`resume`) operator. The symmetric operator `transfer` suspends the currently executing coroutine and executes another². The asymmetric operator `resume` builds

¹ This terminology is due to De Moura and Ierusalimschy [14]. A coroutine is *stackful*, if it can suspend inside nested function calls. Coroutines are *asymmetric* if coroutine activity is organized in a tree-like manner: each coroutine invocation or resumption always returns and yields to its caller. In contrast, *symmetric* coroutines can transfer control among each others without restrictions.

² We use the keywords established by De Moura and Ierusalimschy [14]. In Simula [5], `transfer` corresponds to the system procedure `RESUME`, whereas “asymmetric”, `yield` and `resume` correspond to “semi-symmetric”, `DETACH` and `CALL`, respectively.

$$\begin{array}{ll}
B ::= \text{Bool} \mid \text{Unit} \mid \dots & v ::= k^0 \mid \text{fix } \lambda f. \lambda x. e \mid \ell \\
k^0 ::= \text{true} \mid \text{false} \mid \text{unit} \mid \dots & e ::= k^n e_1 \dots e_n \mid \text{fix } \lambda f. \lambda x. e \\
k^1 ::= \neg \mid \dots & \quad \mid x \mid e e \mid \text{if } e \text{ then } e \text{ else } e \\
k^2 ::= \wedge \mid \dots & \quad \mid \text{create } x. e \mid \text{yield } e \\
\ell, \ell', \dots \in \text{Labels} & \quad \mid \text{resume } e e e e \mid \text{transfer } e e \\
x, y, f, \dots \in \text{Var} & \quad \mid \ell \\
\varphi ::= \perp \mid \tau \rightsquigarrow \tau / \tau \mid \top & \tau ::= B \mid \tau \xrightarrow{\varphi} \tau \mid \top \mid \perp \mid \tau \rightsquigarrow \tau / \tau
\end{array}$$

Fig. 1: Syntax.

a caller-callee relationship: if a coroutine **resumes** another coroutine, they become caller and callee. The **yield** operator inside the callee suspends the coroutine and returns control to the caller. Each of the three operators passes a value. In the remaining paper, we understand “activate” to mean either **transfer** or **resume**, but not **yield**.

The caller-callee relationship is also used when a coroutine finally returns a value, as the value is then passed to the caller. Activating a coroutine after it has returned causes a run-time error; hence, the caller needs to know whether the callee coroutine has terminated. **resume** requires therefore as its third and fourth parameter two *result functions*, one to call with yielded values and one to call with the returned value³.

The language includes a countable set of primitive functions k^n , having each an arity $n \geq 0$. Partial application of primitive functions is not allowed.

2.1 Examples

This section contains short examples of CorDuroy programs. We assume that integers and strings are among the basic types B and that there are constants k^n for arithmetic operations, comparison, and printing. We also use the common `let · = · in ·` sugar for readability.

Divisors. Generators can be used to compute sequences one element at a time. Fig. 2a shows a coroutine which generates the divisors of a number, and a consumer which iterates over the divisors until the generator returns (and the second result function of **resume** is called).

Mutable references. Coroutines are the only stateful construct in CorDuroy. In Fig. 2b, a mutable reference is simulated by a coroutine which keeps an integer value in a local variable. Whenever it is resumed with a function `Int → Int`, it

³ Alternatively, the λ -calculus could be extended with variant types in order to tag the result of **resume** with how it was obtained. We chose the two-continuation **resume** for simplicity.

```

1 let divisors_of = λn.
2   create ..λ..
3     ((fix λloop.λk.
4       if (> k n) then unit
5       else let rem = (mod n k) in
6         let _ = (if (= rem 0)
7           then yield k else unit)
8         in loop (+ k 1)) 1)
9   in let g = divisors_of 24 in
10    ((fix λf. λ..
11      resume g unit
12      (λn. let _ = (print_int n)
13        in (f unit))
14      (λ_. (print_str "finito")))
15      unit)
16 // output: 1 2 3 4 6 8 12 24 finito
(a) Compute all divisors.

1 let makeref = λx0.
2   let main = fix λloop.λx.λupd.
3     let x' = upd x in
4     let upd' = yield x' in
5       loop x' upd'
6   in create .. main x0
7   in let undef = fix λf.λx.(f x)
8   in let write = λr.λv.
9     resume r (λ..v)
10    (λ..unit) (λ..unit)
11   in let read = λ r.
12     resume r (λx.x) (λx.x) undef
13   in
14   let r = makeref 1 in
15   let _ = print_int (read r) in
16   let _ = write r 2 in
17   print_int (read r)
18 // output: 1 2
(b) Mutable references.

```

Fig. 2: Code examples.

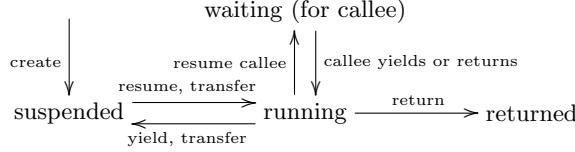


Fig. 3: Life cycle of coroutines.

lets the function update the value and returns the new value. The example also shows how `fix` can be used to create a diverging function with any desired return type in the `read` function⁴.

2.2 Operational Semantics

This section presents a small-step operational semantics for CorDuroy, starting with a life-cycle based view on coroutines to motivate the stack-based representation used in the reduction rules in Fig. 5.

The life cycle of a coroutine consists of the states *suspended*, *running*, *waiting* and *returned*, as shown in Fig. 3. At any moment, there is only one running coroutine.

The running coroutine can apply `create` to a function, creating a new coroutine which starts life in the suspended state (E-CREATE). It can also `resume` a

⁴ The language could alternatively be extended with a special variant of the `resume` operator for coroutines which never return.

$$\begin{aligned}
C ::= & \square \mid k^n v_1 \dots v_{i-1} C e_{i+1} \dots e_n \quad (1 \leq i \leq n) \\
& \mid e C \mid C v \mid \text{if } C \text{ then } e \text{ else } e \\
& \mid \text{resume } C e e e \mid \text{resume } v C e e \mid \text{resume } v v C e \mid \text{resume } v v v C \\
& \mid \text{yield } C \mid \text{transfer } C e \mid \text{transfer } v C
\end{aligned}$$

$$\begin{aligned}
S ::= & \ell @ e; S^? & \text{labels}(\ell @ e; S^?) = \{\ell\} \cup \text{labels}(S^?) \\
S^? ::= & \epsilon \mid S & \text{labels}(\epsilon) = \emptyset
\end{aligned}$$

Fig. 4: Evaluation contexts and stacks.

$$\begin{array}{c}
\frac{n > 0}{\langle \ell @ C[k^n v_1 \dots v_n] ; S^? \mid \mu \rangle \rightarrow \langle \ell @ C[\llbracket k^n \rrbracket(v_1, \dots, v_n)] ; S^? \mid \mu \rangle} \text{E-CONST} \\
\frac{\langle \ell @ C[(\text{fix } \lambda f. \lambda x. e) v] ; S^? \mid \mu \rangle \rightarrow \langle \ell @ C[e[f \mapsto \text{fix } \lambda f. \lambda x. e][x \mapsto v]] ; S^? \mid \mu \rangle} \text{E-FIX} \\
\frac{\langle \ell @ C[\text{if true then } e_t \text{ else } e_f] ; S^? \mid \mu \rangle \rightarrow \langle \ell @ C[e_t] ; S^? \mid \mu \rangle} \text{E-IFT} \\
\frac{\langle \ell @ C[\text{if false then } e_t \text{ else } e_f] ; S^? \mid \mu \rangle \rightarrow \langle \ell @ C[e_f] ; S^? \mid \mu \rangle} \text{E-IFF} \\
\frac{x^* \notin \text{free}(e) \cup \{x\} \quad \ell^* \notin \text{dom}(\mu) \cup \text{labels}(S^?) \cup \{\ell\} \quad v^* = \lambda x^*. ((\lambda x. e x^*) \ell^*) \quad \mu' = \mu \cup \{(\ell^*, v^*)\}} \text{E-CREATE} \\
\frac{\langle \ell @ C[\text{create } x. e] ; S^? \mid \mu \rangle \rightarrow \langle \ell @ C[\ell^*] ; S^? \mid \mu' \rangle} \\
\frac{\ell' \in \text{dom}(\mu) \quad e = \text{resume } \ell' v_a v_s v_n}{\langle \ell @ C[e] ; S^? \mid \mu \rangle \rightarrow \langle \ell' @ (\mu(\ell') v_a) ; \ell @ C[e] ; S^? \mid \mu \setminus \ell' \rangle} \text{E-RES} \\
\frac{x^* \text{ fresh} \quad e_2 = \text{resume } \ell v_a v_s v_n}{\langle \ell_1 @ C_1 [\text{yield } v_y] ; \ell_2 @ C_2 [e_2] ; S^? \mid \mu \rangle \rightarrow \langle \ell_2 @ C_2 [v_s v_y] ; S^? \mid \mu[\ell_1 \mapsto \lambda x^*. C_1[x^*]] \rangle} \text{E-YIE} \\
\frac{\langle \ell_1 @ v_r ; \ell_2 @ C_2 [\text{resume } \ell v_a v_s v_n] ; S^? \mid \mu \rangle \rightarrow \langle \ell_2 @ C_2 [v_n v_r] ; S^? \mid \mu \rangle} \text{E-CORET} \\
\frac{\langle \ell @ C [\text{transfer } \ell v_a] ; S^? \mid \mu \rangle \rightarrow \langle \ell @ C[v_a] ; S^? \mid \mu \rangle} \text{E-TRASELF} \\
\frac{\ell' \in \text{dom}(\mu) \quad x^* \text{ fresh}}{\langle \ell @ C [\text{transfer } \ell' v] ; S^? \mid \mu \rangle \rightarrow \langle \ell' @ (\mu(\ell') v) ; S^? \mid (\mu \setminus \ell') \cup \{(\ell, \lambda x^*. C[x^*])\} \rangle} \text{E-TRA} \\
\frac{\text{E-TRAERR} \quad \ell' \notin \text{dom}(\mu) \quad \text{E-RESERR} \quad \ell' \notin \text{dom}(\mu)}{\langle \ell @ C [\text{transfer } \ell' v_a] \mid \mu \rangle \rightarrow \text{Error} \quad \langle \ell @ C [\text{resume } \ell' v_a v_s v_n] \mid \mu \rangle \rightarrow \text{Error}} \text{E-TRAERR}
\end{array}$$

Fig. 5: Small-step operational semantics rules.

suspended coroutine (the *callee*), becoming its *caller* (E-RES). In doing so, it enters the *waiting* state, and the callee becomes *running*.

A running coroutine can also *yield*, after which it is *suspended* and the caller *running* (E-YIE). If a running coroutine reduces to a value, it is said to *return* that value. The returning coroutine enters its terminal state, and the value is then passed to the caller if there is one (E-CORET) or becomes the final result of the program.

Alternatively, the running coroutine can *transfer* control to a suspended coroutine, suspending itself. In this case, the successor coroutine not only enters the running state, but it also becomes the (only) callee of the predecessor's caller (E-TRA).

In the rules, the state of a program being evaluated is represented as a pair $\langle S \mid \mu \rangle$ of stack and store. The stack S contains, from left to right, the running coroutine, its caller, its caller's caller and so on, each in the form of labeled contexts $\ell@e$ (see Fig. 4). As the running coroutine is the top of the stack, the reduction rules must never pop the last labeled context off the stack.

All suspended coroutines⁵ are kept in the store μ , a function from labels ℓ to values v . The values in the store are the continuations of the yield and transfer expressions which caused the coroutine to be suspended, or, in the case of newly created coroutines, functions which are constructed to be applied likewise.

Coroutines in the returned state are neither in the stack nor in the store because they play no further role in the execution. The waiting and suspended states resemble each other in that neither state permits β -reductions.

The coroutine-related rules all maintain the invariant that a coroutine never rests in the stack and in the store simultaneously. Rule E-CREATE sets up a continuation v^* which makes the new label ℓ^* known under the name x inside the body expression e and passes the first input value to e . E-RES removes the stored continuation of the given coroutine from the store and applies it to the argument in a new labeled context on top of the stack. In the now-waiting coroutine, the *resume* expression remains, awaiting a result from the coroutine above. The third and fourth *resume* parameters are the result functions to be called later with yielded and returned values, respectively.

E-YIE and E-TRA put the continuation of the running coroutine into the store. While E-YIE passes the argument to the first of the two result functions of the caller, E-TRA sets up a new stack top in which the continuation from the store is applied to the argument, just like in E-RES. E-CORET passes the return value to the other result function in the caller and discards the callee. Of the *resume* expression in E-YIE and E-CORET, only the two result functions are used; the old label ℓ need not match the returning or yielding coroutine because the stack top may have been replaced in a *transfer* action.

If a coroutine attempts to activate another coroutine which is not in the store (i.e., not suspended), execution aborts with a run-time error (E-RESERR,

⁵ An implementation would keep the coroutines within the store all the time and annotate them with their state instead; however, the notion of putting coroutines into the store and taking them out again makes the rules easier to read.

E-TRAERR)⁶. As an exception to this rule, a coroutine may safely transfer to itself, e.g. in a multitasking system with just one ready task (E-TRASELF).

Rule E-YIE is only enabled if the stack contains a suitable waiting coroutine below; fortunately, the type system rejects all programs in which a yield expression could appear as a redex in the lowest labeled context.

There is no distinguished main program; the initial expression is also treated as a coroutine, except that it starts in the running state. In order to evaluate a CorDuroy program e , it is wrapped in an initial state with the fixed label ℓ_0 :

$$\text{initState}(e) = \langle S_0 \mid \emptyset \rangle \quad S_0 = \ell_0 @ e; \epsilon \quad (1)$$

The function $\llbracket \cdot \rrbracket$ in E-CONST maps primitive function symbols $k^n, n > 0$ to partial functions of the same arity. The notation $e[x \mapsto f]$ stands for standard capture-avoiding substitution which replaces all free occurrences of x in e by f . The set of free variables in e is $\text{free}(e)$.

2.3 Type system

The type system ensures that values passed to and from coroutines do not cause type errors at run time, and that coroutine operations within the same coroutine body are compatible with each other. It is based on the simply-typed λ -calculus, with an effect system describing which coroutine actions may occur during the evaluation of an expression.

Effects. The effect part of the type and effect system summarizes the yield and transfer expressions which may be evaluated during the evaluation of an expression. The propagation of effects through function application permits a called function to yield and transfer on behalf of the running coroutine in a type-safe way.

If an expression has the effect $\tau_i \rightsquigarrow \tau_o / \tau_r$, then its execution may yield a value of type τ_o to the calling coroutine and expect a value of type τ_i when it is activated again. It may also transfer execution to a coroutine which yields values of type τ_o or returns a value of type τ_r .

Effects φ form a lattice with bottom element \perp and top element \top (see Fig. 6). \perp means that the expression will under no circumstance ever yield. Effect \top means that yield expressions with different types are possible and nothing can be said about the values.

Types. The type system features basic types B , function types, coroutine types as well as top and bottom types.

⁶ This class of runtime errors can be eliminated if E-RES and E-TRA leave the coroutine in the store. Then, activating a terminated or waiting coroutine would invoke (a copy of) the last stored continuation, similar to multi-shot continuations. We chose the error-rules because they are more similar to how Lua and Python handle these situations, and they do not need a facility to copy continuations.

$\begin{aligned}\perp \sqcup \varphi &= \varphi \sqcup \perp = \varphi \\ \top \sqcup \varphi &= \varphi \sqcup \top = \top \\ (\tau_i \rightsquigarrow \tau_o / \tau_r) \sqcup (\tau'_i \rightsquigarrow \tau'_o / \tau'_r) &= (\tau_i \sqcap \tau'_i) \rightsquigarrow (\tau_o \sqcup \tau'_o) / (\tau_r \sqcup \tau'_r) \\ \varphi_1 \sqsubseteq \varphi_2 \quad \text{iff} \quad \exists \varphi'_1. \varphi_1 \sqcup \varphi'_1 = \varphi_2 &\end{aligned}$ <p style="text-align: center;">(a) Effects.</p>	$\begin{aligned}\top \sqcup \tau &= \tau \sqcup \top = \top \\ \perp \sqcap \tau &= \tau \sqcap \perp = \perp \\ \perp \sqcup \tau &= \tau \sqcup \perp = \top \sqcap \tau = \tau \sqcap \top = \tau\end{aligned}$ $\begin{cases} \tau_1^+ & \tau_1^+ = \tau_2^+ \\ \top & \text{otherwise} \end{cases}$ $\begin{cases} \tau_1^+ & \tau_1^+ = \tau_2^+ \\ \perp & \text{otherwise} \end{cases}$ <p style="text-align: center;">(b) Types.</p>
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Fig. 6: Join and meet.

Function arrows are annotated with the effect which may occur during the function's evaluation. We write $\tau_1 \rightarrow \tau_2$ for $\tau_1 \xrightarrow{\perp} \tau_2$.

A value of type $\tau_i \rightsquigarrow \tau_o / \tau_r$ corresponds to a coroutine which can be resumed with values of input type τ_i and yields values of output type τ_o or returns a value of return type τ_r .

Types form a flat lattice with bottom \perp and top \top . For simplicity, subtyping is not allowed, and subeffecting is only allowed in `create` and `fix` expressions. Join and meet on types are defined in figure 6, where τ^+ represent types except for \top and \perp .

Typing rules. The rules are given in Fig. 7. The type environment Γ maps variables to their types. The store typing

$$\Sigma \subseteq \text{Labels} \times \{\tau_i \rightsquigarrow \tau_o / \tau_r \mid \tau_{i,o,r} \neq \top, \tau_i \neq \perp\} \quad (\text{Def}\Sigma)$$

maps labels to the types of the corresponding coroutines at run time. The exclusion of \top and \perp serves to avoid subtyping. Note that type rules do not extend Σ ; expressions are type-checked against a fixed Σ , and preservation (Sec. 3.1) guarantees that some Σ can be found after each evaluation step.

The type function $\text{bastyk}(k^n)$ maps constants to their types of the form $B_1 \rightarrow B_2 \rightarrow \dots \rightarrow B_{n+1}$. We assume that $\text{bastyk}(k^n)$ agrees with the primitive denotation $\llbracket k^n \rrbracket$. We also assume that `true` and `false` are the only k^0 of type `Bool`, and that only `unit` inhabits `Unit`.

Most type rules compute the effect of their expression by joining the effects of the subexpressions. The only exceptions are `T-FIX` and `T-CREATE`, in which the effect of the body expression is moved onto the function arrow or into the coroutine type.

The `create` expression creates a coroutine from a function. In doing so, it binds a variable to the freshly created coroutine label.

`yield` and `transfer` contribute an effect with its input type τ_i . Both suspend the current coroutine and expect a value of type τ_i the next time it is activated. The output and return types in the effect of `yield` describe that `yield` certainly causes the coroutine to yield a value of that type, but never causes a return. `transfer`,

however, transfers control *and the relationship to the caller* to a coroutine which, in turn, may yield and return. Therefore, T-TRA puts the other coroutine’s output and return types into the effect in order to force the surrounding yield and return expressions to match.

Rule T-PROG defines when an entire program is well-typed. The input type Unit is an arbitrary choice, but since the initial label ℓ_0 is not lexically accessible in the program, the input type is of little importance anyway⁷. The output type is bounded to \perp so that an expression which yields can never be the bottom-most expression in a stack (and *yield* with $e : \perp$, while allowed, will diverge instead of yielding).

The initial store typing for a program e with $\vdash_{\text{prog}} e : \tau$ is defined as follows:

$$\Sigma_0^\tau = \{(\ell_0, \text{Unit} \rightsquigarrow \perp/\tau)\} \quad (2)$$

3 Soundness

This section contains the soundness proof. In Sec. 3.1, we prove that reduction steps preserve typing. Sec. 3.2 contains the progress proof, stating that all well-typed execution states are reducible or have finished.

3.1 Preservation

This section states and proves the preservation theorem (Theorem 1). We define the notion of a well-typed execution state before we formulate some lemmas in preparation for the main proof.

Fig. 8 contains the definition of an execution state $\langle S \mid \mu \rangle$ being well-typed, T-STATE. Apart from requiring that the types of store and stack members correspond to the store typing Σ , which is defined in T-STORE and T-STACKN, it poses a constraint $\Sigma \vdash_w S$ about the waiting coroutines in the stack: the redex of waiting callers must be a `resume` expression whose result functions are compatible with the output and return types of the callee.

Lemma 1 *If $\emptyset \mid \Sigma \vdash C[e] : \tau \& \varphi$, then $\emptyset \mid \Sigma \vdash e : \tau' \& \varphi'$ for some τ' , $\varphi' \sqsubseteq \varphi$, and $\text{free}(e) = \emptyset$*

Proof. By induction on C , using that the type rules for all term constructors in C require the subexpressions to be well-typed, that all these type rules join the effects of the subexpressions, and the rules for C never enter the term constructors whose typing rules modify Γ . ■

Lemma 2 *If $\Gamma \mid \Sigma \vdash v : \tau \& \varphi$, then $\varphi = \perp$ and $\tau \neq \perp$.*

⁷ If the program’s design features multiple coroutines transferring to each other, there is still the possibility of having the initial program create one or more such coroutines, each of which knows its label, and transferring control to one of them.

$$\begin{array}{c}
\text{T-CONST} \quad \text{T-VAR} \\
\text{basty}_k(k^n) = B_1 \rightarrow \dots \rightarrow B_{n+1} \quad \forall i = 1 \dots n. \quad \Gamma | \Sigma \vdash e_i : B_i \& \varphi_i \quad \frac{\Gamma(x) = \tau}{\Gamma | \Sigma \vdash x : \tau \& \perp} \\
\hline
\Gamma | \Sigma \vdash k^n e_1 \dots e_n : B_{n+1} \& \bigcup_{i=1 \dots n} \varphi_i \quad \text{T-APP} \\
\frac{\Gamma | \Sigma \vdash e_1 : \tau_2 \xrightarrow{\varphi_3} \tau_1 \& \varphi_1 \quad \Gamma | \Sigma \vdash e_2 : \tau_2 \& \varphi_2}{\Gamma | \Sigma \vdash e_1 e_2 : \tau_1 \& \varphi_1 \sqcup \varphi_2 \sqcup \varphi_3} \\
\text{T-IF} \\
\frac{\Gamma | \Sigma \vdash e_c : \text{Bool} \& \varphi_c \quad \Gamma | \Sigma \vdash e_t : \tau \& \varphi_t \quad \Gamma | \Sigma \vdash e_f : \tau \& \varphi_f}{\Gamma | \Sigma \vdash \text{if } e_c \text{ then } e_t \text{ else } e_f : \tau \& \varphi_c \sqcup \varphi_t \sqcup \varphi_f} \\
\text{T-FIX} \quad \text{T-LABEL} \\
\frac{\Gamma, f : \tau_1 \xrightarrow{\varphi} \tau_2, x : \tau_1 | \Sigma \vdash e : \tau_2 \& \varphi' \quad \varphi' \sqsubseteq \varphi}{\Gamma | \Sigma \vdash \text{fix } \lambda f. \lambda x. e : (\tau_1 \xrightarrow{\varphi} \tau_2) \& \perp} \quad \frac{\Sigma(\ell) = \tau_i \rightsquigarrow \tau_o / \tau_r}{\Gamma | \Sigma \vdash \ell : \tau_i \rightsquigarrow \tau_o / \tau_r \& \perp} \\
\text{T-CREATE} \\
\frac{\Gamma, x : \tau_i \rightsquigarrow \tau_o / \tau_r | \Sigma \vdash e : \tau_i \xrightarrow{\varphi} \tau_r \& \varphi' \quad \varphi, \varphi' \sqsubseteq \tau_i \rightsquigarrow \tau_o / \tau_r \quad \tau_{i,o,r} \neq \top, \tau_i \neq \perp}{\Gamma | \Sigma \vdash \text{create } x.e : \tau_i \rightsquigarrow \tau_o / \tau_r \& \perp} \\
\text{T-RES} \\
\frac{\Gamma | \Sigma \vdash e_c : \tau_i \rightsquigarrow \tau_o / \tau_r \& \varphi_1 \quad \Gamma | \Sigma \vdash e_s : \tau_o \xrightarrow{\varphi_3} \tau_q \& \varphi_4 \quad \Gamma | \Sigma \vdash e_n : \tau_r \xrightarrow{\varphi_5} \tau_q \& \varphi_6}{\Gamma | \Sigma \vdash \text{resume } e_c e_a e_s e_n : \tau_q \& \bigcup_{i=1 \dots 6} \varphi_i} \\
\text{T-YIE} \quad \text{T-TRA} \\
\frac{\Gamma | \Sigma \vdash e : \tau_o \& \varphi_1 \quad \tau_i \neq \top}{\Gamma | \Sigma \vdash \text{yield } e : \tau_i \& (\tau_i \rightsquigarrow \tau_o / \perp) \sqcup \varphi_1} \quad \frac{\Gamma | \Sigma \vdash e_c : \tau_a \rightsquigarrow \tau_o / \tau_r \& \varphi_1 \quad \Gamma | \Sigma \vdash e_a : \tau_a \& \varphi_2}{\Gamma | \Sigma \vdash \text{transfer } e_c e_a : \tau_i \& (\tau_i \rightsquigarrow \tau_o / \tau_r) \sqcup (\varphi_1 \sqcup \varphi_2)} \\
\text{T-PROG} \\
\frac{\emptyset | \emptyset \vdash e : \tau \& \varphi \quad \varphi \sqsubseteq \text{Unit} \rightsquigarrow \perp / \tau}{\vdash_{\text{prog}} e : \tau}
\end{array}$$

Fig. 7: Typing rules.

Proof. Trivial from the definition of v and the typing rules. \blacksquare

Lemma 3 *If $\Gamma, x : \tau' | \Sigma \vdash e : \tau \& \varphi$ and $\emptyset | \Sigma \vdash v : \tau' \& \perp$, then $\Gamma | \Sigma \vdash e[x \mapsto v] : \tau \& \varphi$.*

Proof. By induction over type derivations, replacing T-VAR instances of free x in e with type derivations for v . \blacksquare

Definition 1 *Given Γ, Σ , we write $\Gamma | \Sigma \vdash e_1 \leq e_2$, if $\Gamma | \Sigma \vdash e_1 : \tau \& \varphi_1$ and $\Gamma | \Sigma \vdash e_2 : \tau \& \varphi_2$ with $\varphi_1 \sqsubseteq \varphi_2$. $\Sigma \vdash e_1 \leq e_2$ is an abbreviation for $\emptyset | \Sigma \vdash e_1 \leq e_2$.*

Lemma 4 (Contexts are effect-monotone) *If $\Gamma | \Sigma \vdash e' \leq e$ and for some τ, φ , $\Gamma | \Sigma \vdash C[e] : \tau \& \varphi$, then $\Gamma | \Sigma \vdash C[e'] : \tau \& \varphi'$ for some $\varphi' \sqsubseteq \varphi$, and $\Gamma | \Sigma \vdash C[e'] \leq C[e]$.*

$$\begin{array}{c}
\text{T-STATE} \\
\frac{\Sigma \vdash_{\text{sto}*} \mu \quad \Sigma \vdash_{\text{le}*} S \quad \Sigma \vdash_w S \quad \text{labels}(S) \cap \text{dom } (\mu) = \emptyset}{\Sigma \vdash \langle S \mid \mu \rangle}
\end{array}$$

$$\text{T-STACK0} \quad \frac{\text{T-STACKN} \quad \ell \notin \text{labels}(S^?) \quad \Sigma \vdash_{\text{le}} \ell @ e \quad \Sigma \vdash_{\text{le}*} S^?}{\Sigma \vdash_{\text{le}*} \epsilon \quad \Sigma \vdash_{\text{le}*} S \quad \text{where } S = \ell @ e; S^?}$$

$$\text{T-STACKE} \\
\frac{\Sigma(\ell) = \tau_i \rightsquigarrow \tau_o / \tau_r \quad \emptyset \mid \Sigma \vdash v : \tau_r \& \varphi \quad \varphi \sqsubseteq \tau_i \rightsquigarrow \tau_o / \tau_r}{\Sigma \vdash_{\text{le}} \ell @ e}$$

$$\text{T-WAITN} \\
\frac{S = \ell_1 @ e_1; S' \quad \tau_i \rightsquigarrow \tau_o / \tau_r = \Sigma(\ell_1) \quad S' = \ell_2 @ e_2; S^? \quad e_2 = C[\text{resume } \ell v_a v_s v_n] \quad \emptyset \mid \Sigma \vdash v_s : \tau_o \xrightarrow{\varphi_s} \tau \& \perp \quad \emptyset \mid \Sigma \vdash v_n : \tau_r \xrightarrow{\varphi_n} \tau \& \perp \quad \Sigma \vdash_w S'}{\Sigma \vdash_w S}$$

$$\text{T-WAIT1} \quad \frac{S = \ell @ e; \epsilon \quad \Sigma(\ell) = \tau_i \rightsquigarrow \perp / \tau_r}{\Sigma \vdash_w S} \quad \text{T-STORE} \quad \frac{\mu \text{ is function} \quad \forall (\ell, v) \in \mu : \Sigma \vdash_{\text{sto}} (\ell, v)}{\Sigma \vdash_{\text{sto}*} \mu}$$

$$\text{T-STOREE} \\
\frac{\Sigma(\ell) = \tau_i \rightsquigarrow \tau_o / \tau_r \quad \emptyset \mid \Sigma \vdash v : \tau_i \xrightarrow{\varphi} \tau_r \& \perp \quad \varphi \sqsubseteq \tau_i \rightsquigarrow \tau_o / \tau_r}{\Sigma \vdash_{\text{sto}} (\ell, v)}$$

Fig. 8: Well-typed execution states, stacks, stores.

Proof. By induction on C , using that the type rules for all term constructors in C compute the effect of a composite expression as a join of the effects of the subexpressions, and none of those type rules introduce variable bindings in Γ . ■

Lemma 5 Let $S = \ell @ C[e] ; S^?$, $S' = \ell @ C[e'] ; S^?$ such that $\Sigma \vdash e' \leq e$. Then $\Sigma \vdash_{\text{le}*} S \Rightarrow \Sigma \vdash_{\text{le}*} S'$ and $\Sigma \vdash_w S \Rightarrow \Sigma \vdash_w S'$ hold.

Proof. $\Sigma \vdash_w \ell @ C[e'] ; S^?$ follows from $\Sigma \vdash_w \ell @ C[e] ; S^?$ because the rules only involve the contents of Σ and $S^?$. Furthermore, $\Sigma \vdash_{\text{le}} \ell @ C[e']$ follows from $\Sigma \vdash_{\text{le}} \ell @ C[e]$ by Lemma 4. Finally, the stacks do not differ in the sequence of labels, so that the distinctness property also holds in the second stack. ■

Lemma 6 Let $S = \ell_1 @ e_1; \ell_2 @ C[\text{resume } \ell' v_p v_s v_n] ; S^?$ such that $\Sigma \vdash_{\text{le}*} S$ and $\Sigma \vdash_w S$. Let $\tau_i \rightsquigarrow \tau_s / \tau_n = \Sigma(\ell_1)$. Let v be a value with $\emptyset \mid \Sigma \vdash v : \tau_\alpha \& \perp$ for an $\alpha \in \{s, n\}$. Then, $S' = \ell_2 @ C[v_\alpha v] ; S^?$ satisfies $\Sigma \vdash_{\text{le}*} S'$ and $\Sigma \vdash_w S'$.

Proof. $\Sigma \vdash_w S'$ follows directly from $\Sigma \vdash_w S$ because the rule for \vdash_w does not inspect the topmost expression.

To prove $\Sigma \vdash_{\text{le}*} S'$, we need to prove that

- (a) $\ell_2 \notin \text{labels}(S^?)$,
- (b) $\Sigma \vdash_{\text{le}} \ell_2 @ C[v_\alpha v]$, and

(c) $\Sigma \vdash_{\text{le}^*} S^?$.

Propositions (a, c) follow directly from the assumed $\Sigma \vdash_{\text{le}^*} S$.

To prove proposition (b), it suffices to show that $\Sigma \vdash v_\alpha v \leq \text{resume } \ell' v_a v_s v_n$, which follows from $\Sigma \vdash_w S$ and the assumed type of v . \blacksquare

Lemma 7 *If $\tau_i \rightsquigarrow \tau_o / \tau_r \sqsubseteq \tau'_i \rightsquigarrow \tau'_o / \tau'_r$, then all of the following hold:*

- $\tau_i = \tau'_i$ or $\tau_i = \top$ or $\tau'_i = \perp$
- $\tau_o = \tau'_o$ or $\tau_o = \perp$ or $\tau'_o = \top$
- $\tau_r = \tau'_r$ or $\tau_r = \perp$ or $\tau'_r = \top$

Proof. Trivial from the definition of \sqsubseteq and \sqsupseteq . \blacksquare

Lemma 8 (Well-typed initial states) *Let e be an expression with $\vdash_{\text{prog}} e : \tau$, and $\langle S \mid \mu \rangle = \text{initState}(e)$. Then $\Sigma_0^\tau \vdash \langle S \mid \mu \rangle$.*

Proof. The label disjointness property and $\Sigma \vdash_{\text{sto}*} \mu$ are trivially true because $\mu = \emptyset$. $\Sigma \vdash_w S$ holds because the output type in Equation (1) was chosen as \perp . Finally, $\Sigma \vdash_{\text{le}^*} S$ holds due to the choice of coroutine type in Σ compared to the condition on type and effect of e in T-PROG. \blacksquare

Theorem 1 (Preservation) *If $\Sigma \vdash \langle S \mid \mu \rangle$ and $\langle S \mid \mu \rangle \rightarrow \langle S' \mid \mu' \rangle$, then $\Sigma' \vdash \langle S' \mid \mu' \rangle$ for some $\Sigma' \supseteq \Sigma$.*

Proof. First, we observe that $S \vdash_{\text{sto}*} \mu$ and the label-disjointness condition are preserved in rules E-CONST through E-FIX because these rules only transform the expression in the topmost labeled context without touching the store or the remaining stack. For the remainder, we consider the evaluation rule used to derive the assumption.

Case distinction on the evaluation rule.

- *Case E-CONST:* So $S = \ell @ C[k^n v_1 \dots v_n] ; S^?$ and $S' = \ell @ C[\llbracket k^n \rrbracket(v_1, \dots, v_n)] ; S^?$. We choose $\Sigma' = \Sigma$. Since $\llbracket \cdot \rrbracket$ is assumed to agree with $\text{basty}_k(\cdot)$, $\Sigma \vdash \llbracket k^n \rrbracket(v_1, \dots, v_n) \leq k^n v_1 \dots v_n$ holds and $\Sigma \vdash_{\text{le}} S'$ follows from Lemma 5 and the observation that $\llbracket k^n \rrbracket(v_1, \dots, v_n)$ and $k^n v_1 \dots v_n$ both have effect \perp , which preserves \vdash_w properties from S to S' . ✓
- *Case E-IFT:* So $S = \ell @ C[\text{if true then } e_1 \text{ else } e_2] ; S^?$ and $S' = \ell @ C[e_1] ; S^?$, and $\mu' = \mu$. Choose $\Sigma' = \Sigma$. With Lemma 5, it remains to prove $\Sigma \vdash e_1 \leq \text{if true then } e_1 \text{ else } e_2$, which is obvious from T-IF. ✓
- *Case E-IFF:* Analogously. ✓
- *Case E-FIX:* So $\mu = \mu'$, $S = \ell @ C[(\text{fix } \lambda f. \lambda x. e'') v] ; S^?$ and $S' = \ell @ C[e'] ; S^?$ with $e' = e''[f \mapsto \text{fix } \lambda f. \lambda x. e][x \mapsto v]$. By inversion of T-APP, $\emptyset \mid \Sigma \vdash (\text{fix } \lambda f. \lambda x. e'') : \tau_1 \xrightarrow{\varphi} \tau_2 \& \perp$ and $\emptyset \mid \Sigma \vdash v : \tau_1 \& \perp$ both hold for some τ_1, τ_2, φ . Hence, by inversion of T-FIX, $\Gamma, f : \tau_1 \xrightarrow{\varphi} \tau_2, x : \tau_1 | e'' \vdash \tau_2 : \varphi' \&$ for some $\varphi' \sqsubseteq \varphi$. The proposition follows using Lemma 3 and Lemma 5. ✓

- *Case E-CREATE:* So $S = \ell @ C[\text{create } x.e]; S?$, $S' = \ell @ C[\ell^*]; S?$, and $\mu' = \mu \cup \{(\ell^*, v^*)\}$ with $v^* = \lambda x^*.((\lambda x.e x^*) \ell^*)$. From the assumed $\Sigma \vdash_{le^*} S$, Lemma 1 yields $\emptyset \mid \Sigma \vdash \text{create } x.e : \tau_c \& \varphi_c$ for some τ_c, φ_c . The only rule to derive this is T-CREATE, from which we can conclude that $\varphi_c = \perp$ and $\tau_c = \tau_i \rightsquigarrow \tau_o / \tau_r$. Furthermore, the same rule requires that

$$x : \tau_c \mid \Sigma \vdash e : \tau_i \xrightarrow{\varphi} \tau_r \& \varphi' \quad (3)$$

for some $\varphi, \varphi' \sqsubseteq \tau_i \rightsquigarrow \tau_o / \tau_r$. Choose $\Sigma' = \Sigma \cup \{(\ell^*, \tau_c)\}$, which still is a function due to freshness condition on ℓ^* . Also, the constraints on occurrences of \top and \perp in $\tau_{i,o,r}$, as demanded in (Def Σ), are satisfied by the precondition in T-CREATE. Then $\emptyset \mid \Sigma' \vdash \ell^* : \tau_c \& \perp$ holds, and $\Sigma' \vdash_{le^*} S'$ follows by Lemma 5. $\Sigma' \vdash_w S'$ follows from $\Sigma \vdash_w S$ (using that \vdash_w is obviously monotone in Σ).

$\Sigma' \vdash_{sto^*} \mu'$ requires that μ' is a function (true due to the freshness of ℓ^*), and that v^* has the right type: $\emptyset \mid \Sigma' \vdash v^* : \tau_i \xrightarrow{\varphi^*} \tau_r \& \perp$ for some $\varphi^* \sqsubseteq \tau_i \rightsquigarrow \tau_o / \tau_r$. This follows from (3) by T-APP and T-FIX, observing that all type derivations using Σ also work with its superset Σ' . ✓

- *Case E-YIE:* Then $S = \ell_1 @ e_1; \ell_2 @ e_2; S?$

and $S' = \ell_2 @ e'_2; S?$
with $e_1 = C_1[\text{yield } v_y]$,
 $e_2 = C_2[\text{resume } \ell v_a v_s v_n]$,
 $e'_2 = C_2[v_s v_y]$.

Also $\mu' = \mu \cup \{(\ell_1, e'_1)\}$
with $e'_1 = \lambda x^*.C_1[x^*]$.

We choose $\Sigma' = \Sigma$.

Due to $\Sigma \vdash_{le^*} S$, Σ must contain entries for ℓ_1, ℓ_2 of the form $\Sigma(\ell_k) = \tau_i^k \rightsquigarrow \tau_o^k / \tau_r^k$ for $k = 1, 2$.

Furthermore, by T-STACKN, $\emptyset \mid \Sigma \vdash e_1 : \tau_r^1 \& \varphi_e^1$ and $\emptyset \mid \Sigma \vdash e_2 : \tau_r^2 \& \varphi_e^2$ must hold for some $\varphi_e^1 \sqsubseteq \varphi^1, \varphi_e^2 \sqsubseteq \varphi^2$ (where $\varphi^k = \tau_i^k \rightsquigarrow \tau_o^k / \tau_r^k$).

To prove $\Sigma' \vdash_{le^*} S'$ and $\Sigma' \vdash_w S'$ using Lemma 6,
we need to show $\emptyset \mid \Sigma \vdash v_y : \tau_o^1 \& \perp$ (the rest follows immediately from the assumptions and $\Sigma' = \Sigma$).

Let τ_y, τ_i^* be the types assigned to v_y and $\text{yield } v_y$, respectively, in the type derivation for the assumed $\Sigma \vdash_{le} \ell_1 @ e_1$.

By T-YIE and Lemma 1, we get

$$\tau_i^* \rightsquigarrow \tau_y / \perp \sqsubseteq \tau_i^1 \rightsquigarrow \tau_o^1 / \tau_r^1 \quad (4)$$

By Lemma 7, $\tau_y = \perp$ (impossible: Lemma 2), or $\tau_o^1 = \top$ (contradicting (Def Σ)), or $\tau_y = \tau_o^1$. ✓

To prove $\Sigma \vdash_{sto^*} \mu'$, it remains to prove that μ' is still a function
(by T-STATE, $\ell_1 \notin \text{dom}(\mu)$, so adding ℓ_1 preserves the function property of μ), and that $\emptyset \mid \Sigma \vdash \mu'(\ell_1) : \tau_i^1 \xrightarrow{\varphi} \tau_r^1 \& \perp$ with some $\varphi \sqsubseteq \varphi_1$.

Applying Lemma 7 to (4), we know that $\tau_i^1 = \tau_i^*$
(T-YIE forbids $\tau_i^* = \top$, (Def Σ) forbids $\tau_i^1 = \perp$).

Setting $\Gamma := x^* : \tau_i$, we immediately get $\Gamma | \Sigma \vdash x^* \leq \text{yield } v_y$, and by Lemma 4, $\Gamma | \Sigma \vdash C_1[x^*] \leq e_1$.

Hence, by T-FIX, $\emptyset | \Sigma \vdash \lambda x^*. C_1[x^*] : \tau_i^1 \xrightarrow{\varphi'} \tau_r^1 \& \perp$ for some $\varphi' \sqsubseteq \varphi_1$. ✓

- Case E-CoRET: Then $S = \ell_1 @ v; \ell_2 @ C[\text{resume } \ell v_a v_s v_n]; S^?$ and $S' = \ell_2 @ C[v_n v]; S^?$.

We choose $\Sigma' = \Sigma$.

From $\Sigma \vdash_{\text{le}^*} S$, we know that $\emptyset | \Sigma \vdash v : \tau_r^1 \& \perp$, where $\tau_i^1 \rightsquigarrow \tau_o^1 / \tau_r^1 = \Sigma(\ell_1)$.

The proposition follows immediately using Lemma 6. ✓

- Case E-RES: So $S = \ell_2 @ C[\text{resume } \ell_1 v_a v_s v_n]; S^?$,

and $S' = \ell_1 @ (v_1 v_a); S$

with $v_1 = \mu(\ell_1), \mu' = \mu \setminus \ell_1$.

Furthermore, we know that $\Sigma(\ell_1) = \tau_i \rightsquigarrow \tau_o / \tau_r$ for some τ_i, τ_o, τ_r because $\Sigma \vdash_{\text{le}^*} S$ holds.

We choose $\Sigma' = \Sigma$.

For $\Sigma' \vdash \langle S' | \mu' \rangle$, we need to prove:

- (a) $\Sigma \vdash_{\text{sto}*} \mu'$,
- (b) $\Sigma \vdash_{\text{le}} \ell_1 @ v_1 v_a$,
- (c) $\ell_1 \notin \text{labels}(S')$ (which yields $\Sigma' \vdash_{\text{le}^*} S'$ together with (b)),
- (d) $\Sigma \vdash_w S'$, and
- (e) $\text{labels}(S') \cap \text{dom}(\mu') = \emptyset$.

Proposition (a) follows immediately from μ' being a subset of μ and the assumption $\Sigma \vdash_{\text{sto}*} \mu$. ✓ Proposition (c) is clear from $\Sigma \vdash \langle S | \mu \rangle$. ✓ Proposition (e) is clear because moving ℓ_1 between sets preserves disjointness. ✓

Proposition (b): prove $\emptyset | \Sigma \vdash v_1 v_a : \tau_r \& \varphi_1$ for some $\varphi_1 \sqsubseteq \tau_i \rightsquigarrow \tau_o / \tau_r$.

By assumption $\Sigma \vdash_{\text{sto}*} \mu$, we know about v_1 that $\emptyset | \Sigma \vdash v_1 : \tau_i \xrightarrow{\varphi'_1} \tau_r \& \perp$ holds

with $\varphi'_1 \sqsubseteq \tau_i \rightsquigarrow \tau_o / \tau_r$.

With Lemma 1, Lemma 2 and T-RES, we conclude that $\emptyset | \Sigma \vdash v_a : \tau_i \& \perp$, which yields the desired result using T-APP.

Proposition (d): By Lemma 1, Lemma 2 and T-RES, we know that $\emptyset | \Sigma \vdash v_s : \tau_o \xrightarrow{\varphi_s} \tau_q \& \perp$

and $\emptyset | \Sigma \vdash v_n : \tau_r \xrightarrow{\varphi_n} \tau_q \& \perp$,

which matches the precondition of T-WAITN about v_s and v_n .

The other preconditions follow directly from the assumptions. ✓

- Case E-TRASELF: So $S = \ell @ e_1; S^?$, and $S' = \ell @ e_2; S^?$ with $e_1 = C[\text{transfer } \ell v_a], e_2 = C[v_a], \mu' = \mu$. We choose $\Sigma' = \Sigma$. By Lemma 1 and inversion of T-TRA, $\emptyset | \sigma \vdash v_a : \tau_i \& \perp$ holds for a τ_a with $\tau_a \rightsquigarrow \tau_o / \tau_r = \Sigma(\ell)$. On the other hand, the effect of the transfer-expression as computed in said rule has input type τ_i , so by Lemma 4 and Lemma 7, $\tau_i = \tau_a$. Therefore, $\Sigma \vdash_{\text{le}^*} S'$ follows using Lemma 5.

The other preconditions of $\Sigma \vdash \langle S' | \mu' \rangle$ follow trivially from $\Sigma \vdash \langle S | \mu \rangle$ because store, label sets and the contents of $S^?$ have not been changed in the transition. ✓

- *Case E-TRA*: So $S = \ell_1 @ e_1; S^?$ and $S' = \ell_2 @ (v_2 v_a); S^?$ with $v_2 = \mu(\ell_2)$, $e_1 = C[\text{transfer } \ell_2 v_a]$, $\mu' = (\mu \setminus \ell_2) \cup \{(\ell_1, \lambda x^*. C[x^*])\}$. Furthermore, we know that $\Sigma(\ell_1) = \tau_i^1 \rightsquigarrow \tau_o^1 / \tau_r^1$ for some $\tau_i^1, \tau_o^1, \tau_r^1$ because $\Sigma \vdash_{\text{le}*} S$ holds.
 We choose $\Sigma' = \Sigma$. For $\Sigma' \vdash \langle S' | \mu' \rangle$, we need to prove: (a) $\Sigma \vdash_{\text{sto}*} \mu'$, and (b) $\Sigma \vdash_{\text{le}} \ell_2 @ v_2 v_a$, (c) $\ell_2 \notin \text{labels}(S^?)$ (yields $\Sigma \vdash_{\text{le}*} S'$ with (b)), (d) $\Sigma \vdash_w S'$, and (e) $\text{labels}(S') \cap \text{dom}(\mu') = \emptyset$.
 Proposition (a): since $\Sigma \vdash_{\text{sto}*} \mu$ and $\Sigma \vdash_{\text{sto}*} \mu''$ for any $\mu'' \subseteq \mu$ (which is obvious from the definition), all we need to prove here is

$$\emptyset \mid \Sigma \vdash \lambda x^*. C[x^*] : \tau_i^1 \xrightarrow{\varphi} \tau_r^1 \& \perp \text{ for some } \varphi \sqsubseteq \tau_i^1 \rightsquigarrow \tau_o^1 / \tau_r^1 \quad (5)$$

We prove this by T-FIX. Hence, it suffices to prove $x^* : \tau_i^1 \mid \Sigma \vdash C[x^*] : \tau_r^1 \& \varphi$. By Lemma 1, $\emptyset \mid \Sigma \vdash \text{transfer } \ell_2 v_a : \tau_i \& \varphi_t$ holds for some $\tau_{i,o,r}, \varphi_t$ with

$$\tau_i \rightsquigarrow \tau_o / \tau_r \sqsubseteq \tau_i^1 \rightsquigarrow \tau_o^1 / \tau_r^1 \quad (6)$$

With Lemma 7 and excluding $\tau_i = \top$ and $\tau_i^1 = \perp$ (both impossible by (Def Σ)), we know that $\tau_i = \tau_i^1$. The type judgment of $C[x^*]$ follows by Lemma 4. ✓

Proposition (b): First, $\Sigma(\ell_2) = \tau_i^2 \rightsquigarrow \tau_o^2 / \tau_r^2$ for some $\tau_{i,o,r}^2$ because the transfer expression is well-typed, the only rule to derive which being T-TRA with its precondition on ℓ_2 , derivable using T-LABEL only. In the same instance of T-TRA, v_a is constrained to have type τ_r^2 . From the assumed $\Sigma \vdash_{\text{sto}*} \mu$ follows $\emptyset \mid \Sigma \vdash e_2 : \tau_i^2 \xrightarrow{\varphi'} \tau_r^2 \& \varphi$ for some $\varphi, \varphi' \sqsubseteq (\tau_i^2 \rightsquigarrow \tau_o^2 / \tau_r^2)$. Therefore by T-APP, $\emptyset \mid \Sigma \vdash (e_2 v_a) : \tau_r^2 \& (\varphi \sqcup \varphi')$. ✓.

Propositions (c) follows from the label disjointness condition in $\Sigma \vdash_{\text{le}*} S$: given that $\ell_2 \in \text{dom}(\mu)$, ℓ_2 cannot be in $\text{labels}(S)$, which excludes the possibility of $\ell_1 = \ell_2$ as well as that of ℓ_2 appearing in $S^?$. ✓

Similarly, proposition (e) follows from ℓ_1 and ℓ_2 having switched set memberships. ✓

Proposition (d): $\Sigma \vdash_w S$ holds by assumption. Since in S' , only the topmost labeled context has been replaced, it suffices for $\Sigma \vdash_w S$ to prove that ℓ_2 has at most the same output and return types, i.e. $\tau_o^2 \in \{\perp, \tau_o^1\}$ and $\tau_r^2 \in \{\perp, \tau_r^1\}$. Applying Lemma 7 to (6) (with $\tau_o^2 = \tau_0$ and $\tau_r^2 = \tau_r$, which hold by T-LABEL), we know that $\tau_o^2 = \perp$, or $\tau_o^1 = \top$ (impossible by def. of Σ), or $\tau_o^2 = \tau_o^1$. Similar for τ_r^2, τ_r^1 . ✓

- *Case E-RESERR, E-TRAERR*: Impossible case because the right-hand side of \rightarrow would have a different shape.

End case distinction on the evaluation rule. ■

3.2 Progress

In this section, we prove the progress property. First, we define a language of redexes in Fig. 9, then we show in Lemma 10 that well-typed expressions are either values or redexes embedded in evaluation contexts, which facilitates the main progress theorem, Theorem 2.

$$\begin{aligned}
R ::= & k^n v_1 \dots v_n \mid (\text{fix } \lambda f. \lambda x. e) v \\
& \mid \text{if true then } e_1 \text{ else } e_2 \mid \text{if false then } e_1 \text{ else } e_2 \\
& \mid \text{create } x. e \mid \text{yield } v \mid \text{resume } \ell v \mid \text{transfer } \ell v
\end{aligned}$$

Fig. 9: The language of redexes.

- Lemma 9 (Canonical forms)**
1. If $\Gamma|\Sigma \vdash v : \tau \xrightarrow{\varphi} \tau' \& \varphi'$, then $v = \text{fix } \lambda f. \lambda x. e$ for some f, x, e .
 2. If $\Gamma|\Sigma \vdash v : \text{Bool} \& \varphi'$, then $v = \text{true}$ or $v = \text{false}$.
 3. If $\Gamma|\Sigma \vdash v : \text{Unit} \& \varphi'$, then $v = \text{unit}$.
 4. If $\Gamma|\Sigma \vdash v : \tau_i \rightsquigarrow \tau_o / \tau_r \& \varphi'$, then $v = \ell$ for some $\ell \in \text{dom}(\Sigma)$.

Proof. Of all typing rules, only T-CONST, T-FIX and T-LABEL match values. Of them only T-CONST yields basic types (with a special assumption being made in Sec. 2.3 about `true` and `false` being the only Bool values and `unit` being the only Unit value), only T-LABEL yields coroutine types, and only T-FIX yields function types. ■

- Lemma 10 ($C[R]$ -decomposition)** Let $\emptyset|\Sigma \vdash e : \tau \& \varphi$ for some e, Σ, τ, φ . Then e is a value, or $e = C[R]$ for some C, R .

Proof. By induction over the type derivation.

- Case T-CONST: So $e = k^n e_1 \dots e_n$ for some $n \geq 0$. If all e_i are values v_i , choose $R = k^n v_1 \dots v_n$ and $C = \square$. Otherwise, let $j = \max\{i \mid e_i \text{ is value}\}$. By induction hypothesis, $e_{j+1} = C'[R']$ for some C', R' . Choose $C = k^n e_1 \dots e_j C' e_{j+2} \dots e_n$ and $R = R'$. ✓
- Case T-LABEL, T-FIX: Then e is a value. ✓
- Case T-VAR: Impossible due to $\Gamma = \emptyset$. ✓
- Case T-APP: Then $e = e_1 e_2$. If $e_1 = v_1, e_2 = v_2$, then by inversion of T-APP, v_1 has a type $\tau \xrightarrow{\varphi} \tau'$, so by Lemma 9, $e_1 = \lambda x. e'$ for some x, e' , or $e_1 = \text{fix } \lambda f. \lambda x. e'$ for some f, x, e' . In both cases, $C = \square$ and $R = e$ is a solution. If e_2 is not a value, then by induction hypothesis, $e_2 = C'[R']$. Choose $C = v_1 C'$ and $R = R'$. If e_1 is not a value, either, then by induction hypothesis, $e_1 = C'[R']$. Choose $C = C' e_2$ and $R = R'$. ✓
- Case T-CREATE: Then $R = e$ and $C = \square$. ✓
- Case T-IF, T-RES, T-YIE, T-TRA: In an analogous manner to T-APP. ■

- Theorem 2 (Progress)** Let $\langle S \mid \mu \rangle$ be an evaluation state and Σ a store typing so that $\Sigma \vdash \langle S \mid \mu \rangle$. Then $S = \ell @ v; \epsilon$ for some v, ℓ , or $\langle S \mid \mu \rangle \rightarrow \langle S' \mid \mu' \rangle$ for some S', μ' , or $\langle S \mid \mu \rangle \rightarrow \text{Error}$.

Proof. From $\Sigma \vdash \langle S \mid \mu \rangle$ follows by definition $S = \ell @ e; S^?$, hence $\emptyset|\Sigma \vdash e : \tau \& \varphi$ for some τ, φ . Thus, by Lemma 10, $e = C[R]$ for some C, R , or e is a value. If e is a value v , then either $S^? = \epsilon$, thus $S = \ell @ v; \epsilon$ (final result computed),

or $S^? = \ell_2 @ e_2; S_2^?$. In that case, $\Sigma \vdash_w S$ entails that e_2 must have the shape $C_2[\text{resume } \ell' v_a v_s v_n]$, and E-CORET applies. Thus remains the $e = C[R]$ case, which we prove by case distinction on R : If e is a value, then either the program has finished or E-CORET applies. Otherwise, case distinction on R : (simple cases omitted for brevity):

- *Case $k^n v_1 \dots v_n$:* Then E-CONST applies. ✓
- *Case $\text{if } b \text{ then } e_1 \text{ else } e_2, b \in \{\text{true}, \text{false}\}$:* Then E-IFT or E-IFF applies. ✓
- *Case $(\text{fix } \lambda f. \lambda x. e) v$:* Then E-FIX applies. ✓
- *Case $\text{create } x. e$:* Then E-CREATE applies. ✓
- *Case $\text{yield } v$:* Then by T-YIE, the yield-subexpression must have a nontrivial effect $\varphi' = \tau'_i \rightsquigarrow \tau'_o / \tau'_r$ with $\tau'_o \neq \perp$ ($\tau'_o = \perp$ only if v has type \perp , impossible for values), and by Lemma 4, $\varphi' \sqsubseteq \varphi$ must hold, so that by Lemma 7, the output type in φ must be τ'_o (or \top , but that is impossible due to $\Sigma \vdash_e \ell @ e$ and (Def Σ) forbidding output type \top).
By assumption, $\Sigma \vdash_w S$ holds, which means that since $\tau'_o \neq \perp$, $S^?$ must have the shape $S^? = \ell_2 @ C_2[\text{resume } \ell v_a v_s v_n]; S_2^?$. Therefore, E-YIE applies. ✓
- *Case $\text{resume } \ell' v_2 v_3 v_4$:* Then either E-RES or E-RESERR applies, depending on whether $\ell' \in \text{dom}(\mu)$ holds or not. ✓
- *Case $\text{transfer } \ell' v$:* Then one of E-TRASELF, E-TRA, E-TRAERR applies. ■

4 Related Work

Formalizations of coroutines. De Moura and Ierusalimschy [14] formally define coroutines in an untyped λ -calculus with mutable variables as a model for Lua coroutines. Their interexpressibility results (e.g. transfer in terms of resume/yield) make heavy use of untyped mutable variables; it is yet unclear which of the transformations can be adapted to a statically-typed setting. Their work contains a comprehensive overview of the state of the art in coroutines and related techniques.

Wang and Dahl [18] formalize the control-flow aspects of idealized Simula coroutines. The operational semantics of Belsnes and Østvold [1] also focuses on the control-flow aspects but includes threads and thread-coroutine interaction. Laird [10] presents a process calculus in which the coroutine is the basic building block. Berdine and coworkers [2] define coroutines in their process calculus.

Language design. Languages with parameterless coroutines include Simula [5], Modula-2 [19], and BETA [9]. However, the type systems of these languages need not treat coroutines with much sophistication because the coroutine operations do not pass values.

Some mainstream dynamically-typed languages like Python [17] and Lua [15] pass values to and from coroutines, but without a static type system. C# [13] has static typing and generators (asymmetric coroutines with parameters only for yield), but as the yield-equivalent may only be used lexically inside the generator's body, the type system avoids the complexity involved with stackful coroutines.

Marlin’s ACL [12] is a (statically typed) coroutine extension of Pascal in which coroutines can accept parameters. In analogy to the separation between procedures and functions in Pascal, it features separate syntax for symmetric and asymmetric coroutines. The problem of procedures performing coroutine operations on behalf of the enclosing coroutine is solved by referring to the static block structure, which simplifies the type system at the expense of flexibility.

Haynes and coworkers [7] express coroutines using continuations in Scheme; Harper and colleagues [6] in turn describe a type system for continuations.

Lazy languages like Haskell [16] get asymmetric coroutines for free: a coroutine can be viewed as a transformer of a stream of input values to a stream of output values, which is straightforward to implement using lazy lists. Blazevic [3] produced a more sophisticated monad-based implementation of symmetric coroutines.

5 Conclusion

We presented CorDuroy, a language with type-safe stackful asymmetric and symmetric first-class coroutines, and proved its soundness. CorDuroy constitutes the first provably sound type system for an eager-evaluated language that supports realistic and expressive facilities for coroutines.

One obvious direction of further research is the addition of polymorphism. For subtype polymorphism, (a subset of) C# would be a promising candidate since it already has generators. Parametric polymorphism would likely bring challenges similar to those caused by mutable references.

As this work was inspired by De Moura and Ierusalimschy’s paper [14] in which they present translations between various styles of coroutines, continuations and threads in an untyped setting with mutable variables, it would be interesting to see if the corresponding typed equivalences also hold.

Currently, the operational semantics contains failure rules. Instead, linearity could be introduced to prevent the activation of returned coroutines by keeping track of the coroutine state.

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