Definitions

1. A complete partial order $(M, \leq)$ has a flat ordering iff
   \[ \forall x, y \in M : x \leq y \Rightarrow x = \perp \]

2. A complete lattice $(M, \leq)$ has a flat ordering iff
   \[ \forall x, y \in M : x \leq y \Rightarrow x = \perp \lor x = y \]

3. Let $(M, \leq)$ and $(N, \leq)$ be complete partial orders, and $f : M \to N$. $f$ is
   (a) monotone iff $x \leq y \Rightarrow f(x) \leq f(y)$;
   (b) strict iff $f(\perp) = \perp$.

4. Let $(M, \leq)$ and $(N, \leq)$ be complete lattices, and $f : M \to N$. $f$ is continuous iff $f$ preserves the least upper bound, i.e. for all chains it holds that
   \[ f \left( \bigsqcup_{i \in I} x^{(i)} \right) = \bigsqcup_{i \in I} f(x^{(i)}) \]

Exercise 1

Given functions $f : M \to N$ and $g : N \to P$, which of the following statements are true? Give a proof or a counter example.

For complete partial orders $(M, \leq)$ and $(N, \leq)$:

1. If $(N, \leq)$ has a flat ordering and $f$ is monotone, then $f$ is strict or constant.

2. If $(M, \leq)$ has a flat ordering and $f$ is strict, then $f$ is monotone.

For complete lattices $(M, \leq), (N, \leq)$, and $(P, \leq)$:

1. If in $(M, \leq)$ every chain is stationary and $f$ is monotone, then $f$ is continuous.

2. If $f$ is monotone, then $f$ is strict.

3. If $f$ and $g$ are monotone (continuous, strict), then $f \circ g$ is monotone (continuous, strict).

4. If $f$ is monotone and $(x^{(i)})_{i \in I}$ is a chain in $M$, then $\bigsqcup_{i \in I} f(x^{(i)}) \leq f(\bigsqcup_{i \in I} x^{(i)})$.

5. If $f$ is continuous, then it is also monotone.

Definition

Let $(M, \leq)$ be a complete lattice, and $P : M \to \mathbb{B} = \{true, false\}$ a predicate. $P$ is continuous iff for every chain $(x^{(i)})_{i \in I}$ in $M$ it holds that $P(x^{(i)}) = true$ for all $i \in I$ implies $P(\bigsqcup_{i \in I} x^{(i)}) = true$.

Exercise 2

Let $(M, \leq)$ be a complete lattice, $f : M \to M$ a continuous function, and $P : M \to \mathbb{B}$ a continuous predicate. Prove that

\[ P(\perp) = true \land \forall x \in M : (P(x) = true \Rightarrow P(f(x)) = true) \]

implies

\[ P(lfp(f)) = true \]

where $lfp(f)$ is the smallest fixed point of $f$. 
Exercise 3
Let \((A, \leq)\) and \((G, \leq)\) be partial orders, and \((\alpha, \gamma)\) be a Galois connection between \(A\) and \(G\), i.e. for \(X \in G\) and \(Y \in A\) it holds:

\[ X \leq \gamma(Y) \iff \alpha(X) \leq Y \]

Which of the following statements are true? Give a proof or a counter example.
1. \(\alpha\) monotone
2. \(\gamma\) monotone
3. \(\alpha = \alpha \circ \gamma \circ \alpha\)
4. \(\gamma = \gamma \circ \alpha \circ \gamma\)

Exercise 4
Let \((L, \leq)\) be a complete lattice, and \(f : L \to L\) a monotone function. If \((L, \leq)\) satisfies the ascending chain condition (ACC), then

\[ \text{lfp}(f) = \bigsqcup_n f^{(n)}(\bot) \]

Submission
- Deadline: 18.05.2010, 09:00, per mail to bieniusa@informatik.uni-freiburg.de, or on paper to Annette Bieniusa, Geb. 079, Room 000-14.
- Late submissions will not be marked.
- Do not forget to put your name on the exercise sheet.
- You might want to read up in Appendix A of *Principles of Program Analysis*. 