Definitions
1. A complete partial order \((M, \leq)\) has a flat ordering iff
\[
\forall x, y \in M : x \leq y \Rightarrow x = \bot \lor x = y
\]

2. Let \((M, \leq)\) and \((N, \leq)\) be complete partial orders, and \(f : M \to N\). \(f\) is
   (a) monotone iff \(x \leq y \Rightarrow f(x) \leq f(y)\);
   (b) strict iff \(f(\bot) = \bot\).

3. Let \((M, \leq)\) and \((N, \leq)\) be complete lattices, and \(f : M \to N\). \(f\) is (Scott) continuous iff \(f\) preserves least upper bounds of chains, i.e. for all chains it holds that
\[
f \left( \bigsqcup_{i \in I} x^{(i)} \right) = \bigsqcup_{i \in I} f(x^{(i)})
\]

Exercise 1
Given functions \(f : M \to N\) and \(g : N \to P\), which of the following statements are true? Give a proof or a counter example.

For complete partial orders \((M, \leq)\) and \((N, \leq)\):
   1. If \((N, \leq)\) has a flat ordering and \(f\) is monotone, then \(f\) is strict or constant.
   2. If \((M, \leq)\) has a flat ordering and \(f\) is strict, then \(f\) is monotone.

For complete lattices \((M, \leq), (N, \leq), (P, \leq)\):
   1. If \((M, \leq)\) satisfies the Ascending Chain Condition and \(f\) is monotone, then \(f\) is continuous.
   2. If \(f\) is monotone, then \(f\) is strict.
   3. If \(f\) and \(g\) are monotone (continuous, strict), then \(g \circ f\) is monotone (continuous, strict).
   4. If \(f\) is monotone and \(\langle x^{(i)} \rangle_{i \in I}\) is a chain in \(M\), then \(\bigsqcup_{i \in I} f(x^{(i)}) \leq f(\bigsqcup_{i \in I} x^{(i)})\).
   5. If \(f\) is continuous, then \(f\) is also monotone.

Solution
1. \(\forall x \in M : f\) monotone and \(\bot \leq x \Rightarrow f(\bot) \leq f(x)\). Since \(N\) has a flat ordering, it follows that \(f(\bot) = \bot \lor f(\bot) = f(x)\). This means that \(f\) is either strict \((f(\bot) = \bot)\), or \(f\) is constant, because for every \(x \in M : f(x) = f(\bot)\).

2. Let \(x, y \in M\). Since \(M\) has a flat ordering, it holds that
\[
x \leq y \Rightarrow x = \bot \lor x = y
\]
   As \(f\) is strict, it follows that
\[
f(x) = f(\bot) = \bot \leq f(y) \lor f(x) = f(y)
\]
   Therefore \(f(x) \leq f(y)\), and \(f\) is monotone.
1. Let \( \langle x^{(i)} \rangle_{i \in I} \) be an (arbitrary) chain in \( M \). Construct an ascending chain \( \langle y^{(j)} \rangle_{j \in \mathbb{N}} \) like this: Take \( y^{(0)} = x^{(i)} \) for a \( x^{(i)} \in \langle x^{(i)} \rangle_{i \in I} \). Then

\[
y^{(j+1)} = \begin{cases} 
x^{(i)} & \text{such that } \bigcup_{k=0}^{j} y^{(k)} \leq x^{(i)} \\
y^{(j)} & \text{otherwise}
\end{cases}
\]

\( \Rightarrow \exists \mathbb{ACC} \exists y_0 : y^{(j_0)} = y^{(j_0+1)} \). Hence, \( y^{(j_0)} = \bigcup_{j \in \mathbb{N}} y^{(j)} = \bigcup_{i \in I} x^{(i)} \).

Since \( f \) is monotone: \( f(y^{(0)}) \leq \ldots \leq f(y^{(j_0)}) = \bigcup_{j \in \mathbb{N}} f(y^{(j)}) \), and also,

\[
\bigcup_{j \in \mathbb{N}} f(y^{(j)}) = \bigcup_{i \in I} f(x^{(i)}).
\]

2. Define partial orders \( M = N = (\{\bot, b\}, \leq) \) with \( \bot \leq b \), and \( f(\bot) = f(b) = b \). Then \( f \) is not monotone, but not strict.

3. • Let \( x, y \in M, x \leq y \Rightarrow f(x) \leq f(y) \Rightarrow g(f(x)) \leq g(f(y)) \), as \( f \) and \( g \) are monotone. Hence, \( g \circ f \) is monotone.

• Let \( \langle x^{(i)} \rangle_{i \in I} \) be a chain in \( M \).

\[
g \left( f \left( \bigcup_{i \in I} x^{(i)} \right) \right) = g \left( \bigcup_{i \in I} f \left( x^{(i)} \right) \right) = \bigcup_{i \in I} g \left( f \left( x^{(i)} \right) \right)
\]

Hence, \( g \circ f \) is continuous.

• Let \( \bot_M \in M \). Then, \( f(\bot_M) = \bot_N \) and \( g(f(\bot_M)) = g(\bot_N) = \bot_P \). Hence \( g \circ f \) is strict.

4. It holds that \( x^{(j)} \leq \bigcup_{i \in I} x^{(i)} \) for all \( j \in I \), and because \( f \) is monotone, it follows that

\[
f(x^{(j)}) \leq f\left( \bigcup_{i \in I} x^{(i)} \right) \quad \forall j \in I.
\]

Hence, \( f(\bigcup_{i \in I} x^{(i)}) \) is an upper bound for the chain \( \langle f(x^{(i)}) \rangle_{i \in I} \), and by definition

\[
\bigcup_{i \in I} f(x^{(i)}) \leq f\left( \bigcup_{i \in I} x^{(i)} \right).
\]

5. Let \( x, y \in M \) with \( x \leq y \). Then, \( x \sqcup y = y \). Since \( f \) is continuous, it follows that

\[
f(y) = f(x \sqcup y) = f(x) \sqcup f(y),
\]

and hence \( f(x) \leq f(y) \).

**Definition**

Let \( (M, \leq) \) be a complete lattice, and \( P : M \to \mathbb{B} = \{\text{true}, \text{false}\} \) a predicate. \( P \) is **continuous** iff for every chain \( \langle x^{(i)} \rangle_{i \in I} \) in \( M \) it holds that \( P(x^{(i)}) = \text{true} \) for all \( i \in I \) implies \( P(\bigcup_{i \in I} x^{(i)}) = \text{true} \).

**Exercise 2**

Let \( (M, \leq) \) be a complete lattice, \( f : M \to M \) a continuous function, and \( P : M \to \mathbb{B} \) a continuous predicate. Prove that

\[
P(\bot) = \text{true} \land \forall x \in M : (P(x) = \text{true} \Rightarrow P(f(x)) = \text{true})
\]

implies

\[
P(\mathbf{lfp}(f)) = \text{true}
\]

where \( \mathbf{lfp}(f) \) is the smallest fixed point of \( f \).
Solution

By induction, \( P(f^i(\bot)) = \text{true} \) for all elements in the chain \( \bot \leq f(\bot) \leq \ldots \): The base case is \( P(\bot) = \text{true} \), and the induction step is

\[
P(f^i(\bot)) = \text{true} \Rightarrow P(f(f^i(\bot))) = \text{true} = P(f^{i+1}(\bot))
\]

(4)

\( P \) is continuous, this means that for every chain \( \langle x(i) \rangle_{i \in I} \) in \( M \) it holds that \( P(x(i)) = \text{true} \) for all \( i \in I \) implies \( P(\bigsqcup_{i \in I} x(i)) = \text{true} \). This gives \( P(\bigsqcup_{i \geq 0} f(i)(\bot)) = \text{true} \). The fixed point theorem then gives \( \bigsqcup_{i \geq 0} f(i)(\bot) = \text{lfp}(f) \).

Exercise 3

Let \((A, \leq)\) and \((G, \leq)\) be partial orders, and \((\alpha, \gamma)\) be a Galois connection between \( A \) and \( G \), i.e. for \( X \in G \) and \( Y \in A \) it holds:

\[
X \leq \gamma(Y) \iff \alpha(X) \leq Y
\]

Which of the following statements are true? Give a proof or a counter example.

1. \( \alpha \) monotone
2. \( \gamma \) monotone
3. \( \alpha = \alpha \circ \gamma \circ \alpha \)
4. \( \gamma = \gamma \circ \alpha \circ \gamma \)

Solution

\( \alpha(X) \leq \alpha(X) \) implies \( X \leq \gamma(\alpha(X)) \), and \( \gamma(Y) \leq \gamma(Y) \) implies \( \alpha(\gamma(Y)) \leq Y \).

1. \( X_1 \leq X_2 \Rightarrow X_1 \leq X_2 \leq \gamma(\alpha(X_2)) \Rightarrow \alpha(X_1) \leq \alpha(X_2) \).
2. \( Y_1 \leq Y_2 \Rightarrow \alpha(\gamma(Y_1)) \leq Y_1 \leq Y_2 \Rightarrow \gamma(Y_1) \leq \gamma(Y_2) \).
3. It holds that \( \alpha(\gamma(\alpha(X))) \leq \alpha(X) \) and \( X \leq \gamma(\alpha(\alpha(X))) \). Therefore, \( \alpha(X) \leq \alpha(\gamma(\alpha(X))) \), and we have shown that \( \alpha = \alpha \circ \gamma \circ \alpha \).
4. It holds that \( \gamma(Y) \leq \gamma(\alpha(\gamma(Y))) \) and \( \alpha(\gamma(\alpha(\gamma(Y)))) \leq Y \). Hence, \( \gamma(\alpha(\gamma(Y))) \leq \gamma(Y) \).
And finally, \( \gamma = \gamma \circ \alpha \circ \gamma \).

Exercise 4

Let \((L, \leq)\) be a complete lattice, and \( f : L \to L \) a monotone function. If \((L, \leq)\) satisfies the ascending chain condition (ACC), then

\[
\text{lfp}(f) = \bigsqcup_{n \in \mathbb{N}} f^{(n)}(\bot)
\]

Solution

\( \langle f^{(n)}(\bot) \rangle_{n \in \mathbb{N}} \) is an ascending chain: By definition, \( \bot \leq f(\bot) \), and monotonicity of \( f \) yields \( f^{(i)}(\bot) \leq f^{(i+1)}(\bot) \) for all \( i \in \mathbb{N} \). By ACC, there exists \( n \in \mathbb{N} : f^{(n)}(\bot) = f^{(n+1)}(\bot) \). Hence, \( f^{(n)}(\bot) := l_0 \) is a fixed point.

Let \( l \) be another fixed point, i.e. \( l = f(l) \). As \( \bot \leq l \) and by monotonicity of \( f \), it holds that

\[
f^{(i)}(\bot) \leq f^{(i)}(l) = l \quad \forall i \in \mathbb{N}.
\]

Therefore, \( l_0 \leq l \), and \( l_0 \) is lfp.