Static Program Analysis

http://proglang.informatik.uni-freiburg.de/teaching/programanalysis/2014ss/

Solution Sheet 3

15.05.2014

Exercise 1 (Posets)

1. Show that for the two partially ordered sets (posets) $(\mathcal{P}(M), \subseteq)$ and $(\mathcal{P}(N), \subseteq)$ the product of the two posets is a poset

$$(\mathcal{P}(M) \times \mathcal{P}(N), \sqsubseteq).$$

The partial order $\sqsubseteq$ is defined as

$$(m_1, n_1) \sqsubseteq (m_2, n_2) \Leftrightarrow m_1 \subseteq m_2 \wedge n_1 \subseteq n_2.$$

You can assume that M and N are disjoint.

2. a) Let $(P_1, \sqsubseteq_1), \ldots, (P_n, \sqsubseteq_n)$ be posets. Show that the cartesian product $P_1 \times \cdots \times P_n$ and the relation $\sqsubseteq^n$, where $\sqsubseteq^n$ is defined as

$$(x_1, \ldots, x_n) \sqsubset^n (y_1, \ldots, y_n) \stackrel{def}{=} \exists i \in [1, n] : \forall j < i : x_j = y_j \wedge x_i \sqsubset_i y_i$$
$$(x_1, \ldots, x_n) \sqsubseteq^n (y_1, \ldots, y_n) \stackrel{def}{=} (x_1, \ldots, x_n) \sqsubset^n (y_1, \ldots, y_n) \vee \bigwedge_{i=1}^{n} x_i = y_i,$$

is a poset.

b) Show that $(P_1 \times \cdots \times P_n, \sqsubseteq^n)$ is totally ordered if $(P_1, \sqsubseteq_1), \ldots, (P_n, \sqsubseteq_n)$ are totally ordered.

c) What is the (unique) top/bottom element $\top/\bot$ of $(P_1 \times \cdots \times P_n, \sqsubseteq^n)$?

d) What requirement(s) on $(P_1, \sqsubseteq_1), \ldots, (P_n, \sqsubseteq_n)$ need to be satisfied for $\top/\bot$ to exist in $(P_1 \times \cdots \times P_n, \sqsubseteq^n)$?

Solution

1. Reflexivity:

Proof. $(n, m) \sqsubseteq (n, m) = m \subseteq m \wedge n \subseteq n$ □

Antisymmetry:

Proof.

$$\begin{aligned} &(n_1, m_1) \sqsubseteq (n_2, m_2) \wedge (n_2, m_2) \sqsubseteq (n_1, m_1) \\ \Longrightarrow\ & m_1 \subseteq m_2 \wedge m_2 \subseteq m_1 \wedge n_1 \subseteq n_2 \wedge n_2 \subseteq n_1 \\ \Longrightarrow\ & (n_1, m_1) = (n_2, m_2) \end{aligned}$$

□

Transitivity:

Proof.

$$
\begin{aligned}
& (n_1, m_1) \sqsubseteq (n_2, m_2) \wedge (n_2, m_2) \sqsubseteq (n_3, m_3) \\
\implies & m_1 \subseteq m_2 \wedge n_1 \subseteq n_2 \wedge m_2 \subseteq m_3 \wedge n_2 \subseteq n_3 \\
\implies & m_1 \subseteq m_3 \wedge n_1 \subseteq n_3 \\
\implies & (n_1, m_1) \sqsubseteq (n_2, m_2)
\end{aligned}
$$

□

2. Note that the order $\sqsubseteq^n$ is the lexicographical order.
Let $\overline{a} = (a_1 \times \cdots \times a_n)$ and $\overline{b} = (b_1 \times \cdots \times b_n)$.
   a) Reflexivity:

$$\overline{a} \sqsubseteq^n \overline{a}$$

Proof. Immediate from the definition of $\sqsubseteq^n$. □

Antisymmetry:

$$\overline{a} \sqsubseteq^n \overline{b} \wedge \overline{b} \sqsubseteq^n \overline{a} \implies \overline{a} = \overline{b}$$

Proof.

Case $(\overline{a} \sqsubset^n \overline{b} \wedge \overline{b} \sqsubset^n \overline{a})$**.** ↯
In particular from $\exists i \in [1, n] : \forall j < i : a_j = b_j \wedge a_i \sqsubset_i b_i$ *follows that* $\nexists i' \in [1, n] : \forall j' < i' : b_{j'} = a_{j'} \wedge b_{i'} \sqsubset_{i'} a_{i'}$ *because* $\forall i \in [1, n] : (a_i = b_i \implies a_i \not\sqsubset_i b_i \wedge b_i \not\sqsubset_i a_i) \wedge (a_i \sqsubset_i b_i \implies a_i \neq b_i)$

Case $(\overline{a} = \overline{b})$**.**

□

Transitivity: $\overline{a} \sqsubseteq^n \overline{b} \wedge \overline{b} \sqsubseteq^n \overline{c} \implies \overline{a} \sqsubseteq^n \overline{c}$

Proof.

Case $(\overline{a} \sqsubset^n \overline{b} \wedge \overline{b} \sqsubset^n \overline{c})$**.** *We know that*

$$\exists i \in [1, n] : \forall j < i : a_j = b_j \wedge a_i \sqsubset_i b_i$$

,

$$\exists i' \in [1, n] : \forall j' < i' a_{j'} = b_{j'} \wedge a_{i'} \sqsubset_{i'} b_{i'}.$$

- *If* $i = i'$ *then because of transitivity of* $\sqsubseteq_i$ *it holds that* $a_i \sqsubset_i c_i$ *such that* $\forall j < i : a_j = b_j \wedge a_i \sqsubset_i c_i$.
- *If* $i \neq i'$ *then it holds that* $\forall j < i'' : a_j = b_j \wedge a_{i''} \sqsubset_{i''} c_{i''}$ *where* $i'' = min(i, i')$.
- *If* $i \neq i'$ *then* $\forall j < i' : a_j = b_j \wedge a_{i'} \sqsubset_{i'} c_{i'}$.

Case $(\overline{a} \sqsubset^n \overline{b} \wedge \overline{b} = \overline{c})$**.** *Immediate.*

Case $(\overline{a} = \overline{b} \wedge \overline{b} \sqsubset^n \overline{c})$**.** *Immediate.*

Case $(\overline{a} = \overline{b} \wedge \overline{b} = \overline{c})$**.** *Immediate.*

□

b) We need to show that $\forall \overline{a}, \overline{b} \in P_1 \times \cdots \times P_n : \overline{a} \sqsubseteq^n \overline{b} \vee \overline{b} \sqsubseteq^n \overline{a}$.

Proof by construction.

Case $(\overline{a} = \overline{b})$**.** *Immediate.*

Case $(\overline{a} \neq \overline{b})$**.** *We use the following algorithm to find* $i \in [1, n]$ *such that* $\forall j < i : a_j = b_j \wedge (a_i \sqsubset_i b_i \vee b_i \sqsubset_i a_i)$.

1. *i=0;*
2. *Due to the the total ordering of* $(P_1, \sqsubseteq_1), \ldots, (P_n, \sqsubseteq_n)$ *we know that either* $a_i \sqsubset_i b_i$, $b_i \sqsubset_i a_i$ *or* $a_i = b_i$.
   If $a_i \sqsubset_i b_i$ *or* $b_i \sqsubset_i a_i$ *i has been found.*
   Otherwise $a_i = b_i$
   A. *i:= i+1;*
   B. *Goto 2.*

The algorithm terminates as $\overline{a} \neq \overline{b}$.
It is easy to see that $\forall j < i : a_j = b_j \wedge (a_i \sqsubset_i b_i \vee b_i \sqsubset_i a_i)$.

□

c) $\top^n = (\top^1 \times \cdots \times \top^n)$ and $\bot^n = (\bot^1 \times \cdots \times \bot^n)$

d) If $\top^1 \ldots \top^n$ exist then $\top^n$ exists. If $\bot^1 \ldots \bot^n$ exist then $\bot^n$ exists.