Compiler Construction 2016/2017
Loop Optimizations

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Outline

Loops

- Loops are everywhere

$\Rightarrow$ worthwhile target for optimization

Loops

Definition: Loop

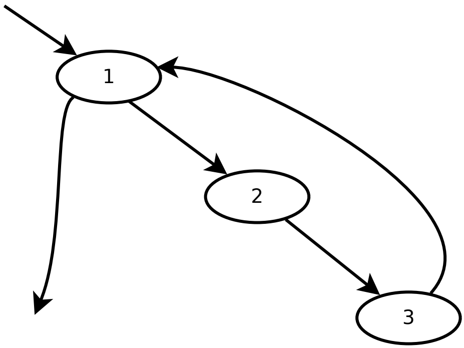
A loop with header h is a set L of nodes in a CFG such that

- $h \in L$
- $(\forall s \in L)$ exists path from h to s
- $(\forall s \in L)$ exists path from s to h
- $(\forall t \notin L)$ $(\forall s \in L)$ if there is an edge from t to s, then $s = h$

Special loop nodes

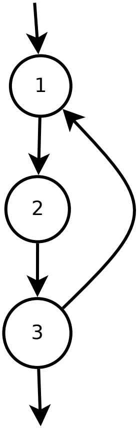
- A loop entry node has a predecessor outside the loop.
- A loop exit node has a successor outside the loop.

Example Loops



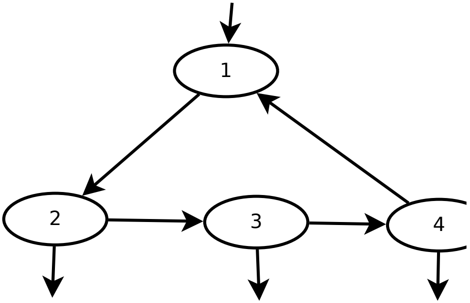
Example Loops

18-1a



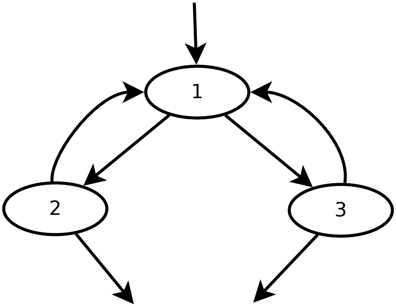
Example Loops

18-1b



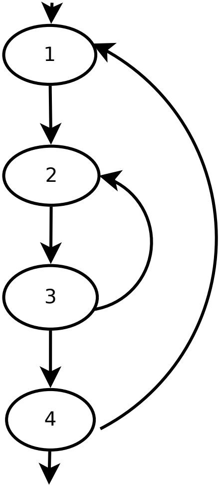
Example Loops

18-1c



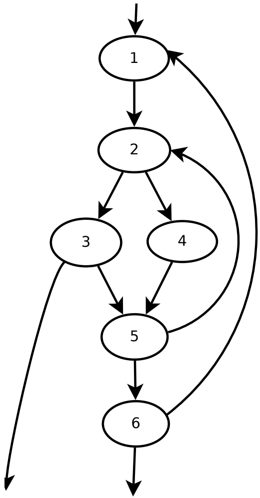
Example Loops

18-1d



Example Loops

18-1e



Program for 18-1e

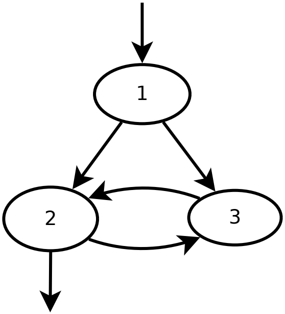
```
int isPrime (int n) {
  i = 2;
  do {
    j = 2;
    do {
      if (i*j==n) {
        return 0;
      } else {
        j = j+1;
      }
    } while (j<n);
    i = i+1;
  } while (i<n);
  return 1;
}
```

Reducible Flow Graphs

- Arbitrary flow graphs: Spaghetti code
- Reducible flow graphs arise from structured control
  - if-then-else
  - while-do
  - repeat-until
  - for
  - break (multi-level)

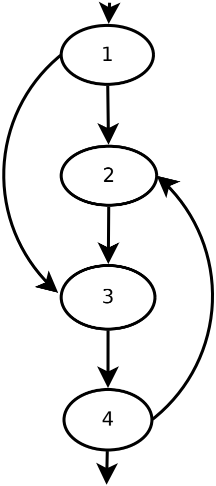
Irreducible Flow Graphs

18-2a: Not a loop



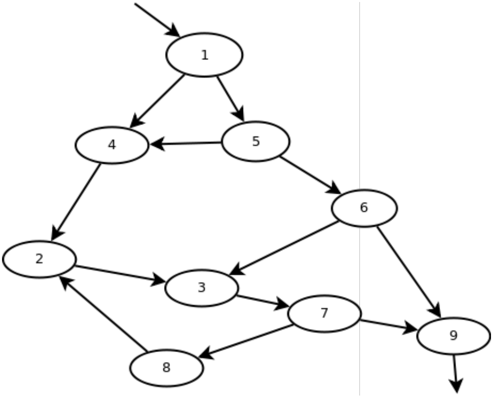
Irreducible Flow Graphs

18-2b: Not a loop



Irreducible Flow Graphs

18-2c: Not a loop



- Reduces to 18-2a: collapse edges (x, y) where x is the only predecessor of y
- A flow graph is irreducible if exhaustive collapsing leads to a subgraph like 18-2a.

Outline

Dominators

Objective

Find all loops in flow graph

Assumption

Each CFG has unique entry node s_0 without predecessors

Domination relation

A node *d* dominates a node *n* if every path from s_0 to *n* must go through *d*.

Remark

Domination is reflexive

Algorithm for Finding Dominators

Lemma

Let n be a node with predecessors $p_1, \ldots, p_k$ and $d \neq n$ a node. d dominates n iff $(\forall 1 \leq i \leq k)$ d dominates p_i

Domination equation

Let $D[n]$ be the set of nodes that dominate n.

$$D[n] = \{n\} \cup \bigcap_{p \in pred[n]} D[p]$$

- Solve by fixed point iteration
- Start with $(\forall n \in N)$ $D[n] = N$ (all nodes in the CFG)
- Observe that $D[s_0] = \{s_0\}$ because $pred(s_0) = \emptyset$
- Watch out for unreachable nodes

Immediate Dominators

Theorem

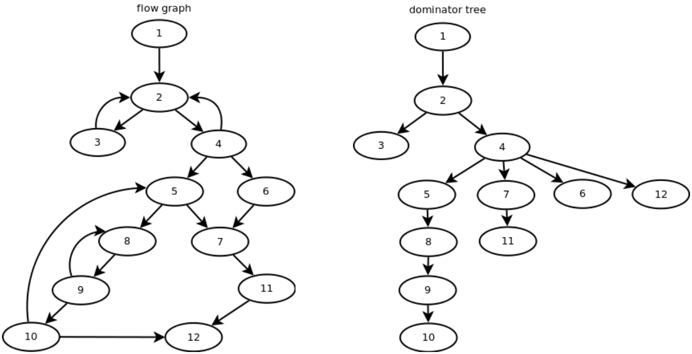
Let G be a connected, rooted graph. If d dominates n and e dominates n, then either d dominates e or e dominates d.

- **Proof:** by contradiction
- **Consequence:** Each node $n \neq s_0$ has one immediate dominator $idom(n)$ such that
  1. $idom(n) \neq n$
  2. $idom(n)$ dominates n
  3. $idom(n)$ does not dominate another dominator of n

Dominator Tree

Dominator Tree

The dominator tree is a directed graph where the nodes are the nodes of the CFG and there is an edge (x, y) if $x = idom(y)$.



- **back edge** in CFG: from n to h so that h dominates n

Finding Loops

Natural Loop

The natural loop of a back edge (n, h) where h dominates n is the set of nodes x such that

- h dominates x
- exists path from x to n not containing h

h is the header of this natural loop.

Nested Loops

Nested Loop

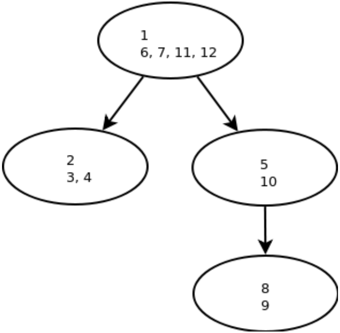
If A and B are loops with headers $a \neq b$ and $b \in A$, then $B \subseteq A$. Loop B is nested within A. B is the inner loop.

Algorithm: Loop-nest Tree

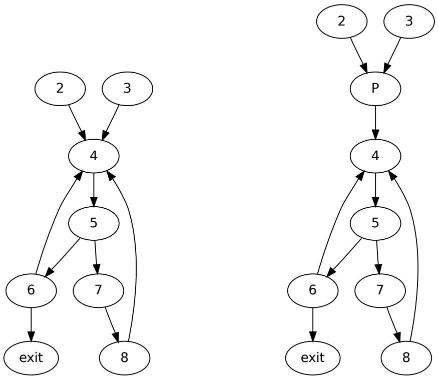
1. Compute the dominators of the CFG
2. Compute the dominator tree
3. Find all natural loops with their headers
4. For each loop header h merge all natural loops of h into a single loop $loop[h]$
5. Construct the tree of loop headers such that h_1 is above h_2 if $h_2 \in loop[h_1]$

- Leaves are innermost loops
- Procedure body is pseudo-loop at root of loop-nest tree

A Loop-Nest Tree



Adding a Loop Preheader



- loop optimizations need a CFG node <u>before the loop</u> as a target to move code out of the loop

$\Rightarrow$ add <u>preheader node</u> like *P* in example

Outline

Loop-Invariant Computations

- Suppose $t \leftarrow a \oplus b$ occurs in a loop.
- If a and b have the same value for each iteration of the loop, then t always gets the same value.

$\Rightarrow$ t's definition is loop-invariant, but its computation is repeated on each iteration

Goals

- Detect such loop-invariant definitions
- Hoist them out of the loop

Approximation to Loop-Invariance

Loop-Invariance

The definition $d : t \leftarrow a_1 \oplus a_2$ is loop-invariant for loop L if $d \in L$ and, for each a_i, one of the following conditions holds:

1. a_i is a constant,
2. all definitions of a_i that reach d are outside of L, or
3. only one definition of a_i reaches d and that definition is loop-invariant.

Algorithm: Loop-Invariance

1. Identify all definitions whose operands are constant or defined outside the loop
2. Add loop-invariant definitions until a fixed point is reached

Hoisting

- Suppose $t \leftarrow a \oplus b$ is loop-invariant.
- Can we hoist it out of the loop?

L_0 $t \leftarrow 0$ L_1 $i \leftarrow i + 1$ $t \leftarrow a \oplus b$ $M[i] \leftarrow t$ if $i < N$ goto L_1 L_2 $x \leftarrow t$	L_0 $t \leftarrow 0$ L_1 if $i \geq N$ goto L_2 $i \leftarrow i + 1$ $t \leftarrow a \oplus b$ $M[i] \leftarrow t$ goto L_1 L_2 $x \leftarrow t$	L_0 $t \leftarrow 0$ L_1 $i \leftarrow i + 1$ $t \leftarrow a \oplus b$ $M[i] \leftarrow t$ $t \leftarrow 0$ $M[j] \leftarrow t$ if $i < N$ goto L_1 L_2	L_0 $t \leftarrow 0$ L_1 $M[j] \leftarrow t$ $i \leftarrow i + 1$ $t \leftarrow a \oplus b$ $M[i] \leftarrow t$ if $i < N$ goto L_1 L_2 $x \leftarrow t$

Hoisting

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yes	*no*	*no*	*no*

Hoisting

Criteria for hoisting

A loop-invariant definition $d : t \leftarrow a \oplus b$ can be hoisted to the end of its loop's preheader if all of the following hold

1. d dominates all loop exits at which t is live-out
2. there is only one definition of t in the loop
3. t is not live-out at the loop preheader

- Attention: arithmetic exceptions, side effects of $\oplus$
- Condition 1 often prevents hoisting from while loops: transform into repeat-until loops.

Outline

Induction Variables

C-code for summation of a **long** array

```
long sum(long a[], int n) {
  long s = 0;
  int i = 0;
  while (i<n) {
    s += a[i];
    i ++;
  }
  return s;
}
```

Induction Variables and Strength Reduction

Consider the corresponding IR

$$
\begin{array}{rl}
 & s \leftarrow 0 \\
 & i \leftarrow 0 \\
L_1: & \text{if } i \geq n \text{ goto } L_2 \\
 & j \leftarrow i \cdot 4 \\
 & k \leftarrow j + a \\
 & x \leftarrow M[k] \\
 & s \leftarrow s + x \\
 & i \leftarrow i + 1 \\
 & \text{goto } L_1 \\
L_2 &
\end{array}
$$

Induction Variables and Strength Reduction

Consider the corresponding IR

$$
\begin{array}{rlcl}
 & s & \leftarrow & 0 \\
 & i & \leftarrow & 0 \\
L_1: & \multicolumn{3}{l}{\text{if } i \geq n \text{ goto } L_2} \\
 & j & \leftarrow & i \cdot 4 \\
 & k & \leftarrow & j + a \\
 & x & \leftarrow & M[k] \\
 & s & \leftarrow & s + x \\
 & i & \leftarrow & i + 1 \\
 & \multicolumn{3}{l}{\text{goto } L_1} \\
L_2 & & &
\end{array}
$$

before

$$
\begin{array}{rlcl}
 & s & \leftarrow & 0 \\
 & k' & \leftarrow & a \\
 & b & \leftarrow & n \cdot 4 \\
 & c & \leftarrow & a + b \\
L_1: & \multicolumn{3}{l}{\text{if } k' \geq c \text{ goto } L_2} \\
 & x & \leftarrow & M[k'] \\
 & s & \leftarrow & s + x \\
 & k' & \leftarrow & k' + 4 \\
 & \multicolumn{3}{l}{\text{goto } L_1} \\
L_2 & & &
\end{array}
$$

after

Induction Variables

- **Induction-variable analysis:**
  identify induction variables and relations among them
- **Strength reduction:**
  replace expensive operation (e.g., multiplication) by cheap operation (e.g., addition)
- **Induction-variable elimination:**
  remove dependent induction variables

Induction Variables

- A basic induction variable is directly incremented
- A derived induction variable is computed from other induction variables
- Describe an induction variable b' by a triple (b, o, f), where
  - b is a basic induction variable
  - o is an offset
  - f is a factor

  so that $b' = o + f \cdot b$.
- A linear induction variable changes by the same amount in every iteration.

Induction Variables in the Example

- i is a basic induction variable described by $(i, 0, 1)$
- j is a derived induction variable:
  after $j \leftarrow i \cdot 4$, it is described by $(i, 0, 4)$
- k is a derived induction variable:
  after $k \leftarrow j + a$, it is described by $(i, a, 4)$

Non-linear Induction Variables

$$
\begin{array}{rlcl}
 & s & \leftarrow & 0 \\
L_1: & \text{if } s > 0 \text{ goto } L_2 & & \\
 & i & \leftarrow & i + b \\
 & j & \leftarrow & i \cdot 4 \\
 & x & \leftarrow & M[j] \\
 & s & \leftarrow & s - x \\
 & \text{goto } L_1 & & \\
L_2: & i & \leftarrow & i + 1 \\
 & s & \leftarrow & s + j \\
 & \text{if } i < n \text{ goto } L_1 & &
\end{array}
$$

Non-linear Induction Variables

$$
\begin{array}{rlll}
 & s & \leftarrow & 0 \\
L_1: & \multicolumn{3}{l}{\text{if } s > 0 \text{ goto } L_2} \\
 & i & \leftarrow & i + b \\
 & j & \leftarrow & i \cdot 4 \\
 & x & \leftarrow & M[j] \\
 & s & \leftarrow & s - x \\
 & \multicolumn{3}{l}{\text{goto } L_1} \\
L_2: & i & \leftarrow & i + 1 \\
 & s & \leftarrow & s + j \\
 & \multicolumn{3}{l}{\text{if } i < n \text{ goto } L_1}
\end{array}
$$

before

$$
\begin{array}{rlll}
 & s & \leftarrow & 0 \\
 & j' & \leftarrow & i \cdot 4 \\
 & b' & \leftarrow & b \cdot 4 \\
 & n' & \leftarrow & n \cdot 4 \\
L_1: & \multicolumn{3}{l}{\text{if } s > 0 \text{ goto } L_2} \\
 & j' & \leftarrow & j' + b' \\
 & j & \leftarrow & j' \\
 & x & \leftarrow & M[j] \\
 & s & \leftarrow & s - x \\
 & \multicolumn{3}{l}{\text{goto } L_1} \\
L_2: & j' & \leftarrow & j' + 4 \\
 & s & \leftarrow & s + j \\
 & \multicolumn{3}{l}{\text{if } j' < n' \text{ goto } L_1}
\end{array}
$$

after

Detection of Induction Variables

Basic Induction Variable (in the family of i)

Variable i is a basic induction variable if all definitions of i in loop L have the form $i \leftarrow i \pm c$ where c is loop-invariant.

Derived Induction Variable

Variable k is a derived ind. var. in the family of i in loop L if

1. there is exactly one definition of k in L of the form $k \leftarrow j \cdot c$ or $k \leftarrow j + d$ where j is an induction variable in the family of i and c, d are loop-invariant
2. if j is a derived induction variable in the family of i, then
   - only the definition of j in L reaches (the definition of) k
   - there is no definition of i on any path between the definition of j and the definition of k
3. If j is described by (i, a, b), then k is described by $(i, a \cdot c, b \cdot c)$ or $(i, a + d, b)$, respectively.

Strength Reduction

- Often multiplication is more expensive than addition

$\Rightarrow$ Replace the definition $j \leftarrow i \cdot c$ of a derived induction variable by an addition

Procedure

- For each derived induction variable $j \sim (i, a, b)$ create new variable j'
- After each assignment $i \leftarrow i + c$ to a basic induction variable, create an assignment $j' \leftarrow j' + c \cdot b$
- Replace assignment to j with $j \leftarrow j'$
- Initialize $j' \leftarrow a + i \cdot b$ at end of preheader

Example Strength Reduction

Induction Variables $j \sim (i, 0, 4)$ and $k \sim (i, a, 4)$

$$
\begin{array}{rlcl}
 & s & \leftarrow & 0 \\
 & i & \leftarrow & 0 \\
L_1: & \multicolumn{3}{l}{\text{if } i \geq n \text{ goto } L_2} \\
 & j & \leftarrow & i \cdot 4 \\
 & k & \leftarrow & j + a \\
 & x & \leftarrow & M[k] \\
 & s & \leftarrow & s + x \\
 & i & \leftarrow & i + 1 \\
 & \multicolumn{3}{l}{\text{goto } L_1} \\
L_2 & & &
\end{array}
$$

before

$$
\begin{array}{rlcl}
 & s & \leftarrow & 0 \\
 & i & \leftarrow & 0 \\
 & j' & \leftarrow & 0 \\
 & k' & \leftarrow & a \\
L_1: & \multicolumn{3}{l}{\text{if } i \geq n \text{ goto } L_2} \\
 & j & \leftarrow & j' \\
 & k & \leftarrow & k' \\
 & x & \leftarrow & M[k] \\
 & s & \leftarrow & s + x \\
 & i & \leftarrow & i + 1 \\
 & j' & \leftarrow & j' + 4 \\
 & k' & \leftarrow & k' + 4 \\
 & \multicolumn{3}{l}{\text{goto } L_1} \\
L_2 & & &
\end{array}
$$

after

Elimination

- Apply constant propagation, copy propagation, and dead code elimination
- Special case: elimination of induction variables that are
  - not used in the loop
  - only used in comparisons with loop-invariant variables
  - useless

Useless variable

A variable is <u>useless</u> in a loop L if

- it is dead at all exits from L
- it is only used in its own definitions

Example After removal of j, j' is useless

Rewriting Comparisons

Almost useless variable

A variable is almost useless in loop L if

- it is only used in comparisons against loop-invariant values and in definitions of itself and
- there is another induction variable in the same family that is not useless.

- An almost useless variable can be made useless by rewriting the comparisons to use the related induction variable

Rewriting Comparisons

Coordinated induction variables

Let $x \sim (i, a_x, b_x)$ and $y \sim (i, a_y, b_y)$ be induction variables. x and y are coordinated if

$$(x - a_x)/b_x = (y - a_y)/b_y$$

throughout the execution of the loop, except during a sequence of statements of the form $z_i \leftarrow z_i + c_i$ where c_i is loop-invariant.

Rewriting Comparisons

Let $j \sim (i, a_j, b_j)$ and $k \sim (i, a_k, b_k)$ be coordinated induction variables.
Consider the comparison $k < n$ with n loop-invariant.
Using $(j - a_j)/b_j = (k - a_k)/b_k$ the comparison can be rewritten as follows

$$
\begin{aligned}
& b_k(j - a_j)/b_j + a_k < n \\
\Leftrightarrow \quad & \\
& b_k(j - a_j)/b_j < n - a_k \\
\Leftrightarrow \quad & \\
& \begin{cases} j < (n - a_k)b_j/b_k + a_j & \text{if } b_j/b_k > 0 \\ j > (n - a_k)b_j/b_k + a_j & \text{if } b_j/b_k < 0 \end{cases}
\end{aligned}
$$

where the right-hand sides are loop-invariant and their computation can be hoisted to the preheader.

Rewriting Comparisons

Restrictions

1. $(n - a_k)b_j$ must be a multiple of b_k
2. b_j and b_k must both be constants or loop invariants of known sign

Outline

Array-Bounds Checks

- Safe programming languages check that the subscript is within the array bounds at each array operation.
- Bounds for an array have the form $0 \leq i < N$ where $N > 0$ is the size of the array.
- Implemented by $i <_u N$ (unsigned comparison).
- Bounds checks redundant in well-written programs $\Rightarrow$ slowdown
- For better performance: let the compiler prove which checks are redundant!
- In general, this problem is undecidable.

Assumptions for Bounds Check Elimination in Loop L

1. There is an induction variable j and loop-invariant u used in statement s_1 of either of the forms
   - if $j < u$ goto L_1 else goto L_2
   - if $j \geq u$ goto L_2 else goto L_1
   - if $u > j$ goto L_1 else goto L_2
   - if $u \geq j$ goto L_2 else goto L_1

   where L_2 is out of the loop L.
2. There is a statement s_2 of the form
   - if $k <_u n$ goto L_3 else goto L_4

   where k is an induction variable coordinated with j, n is loop-invariant, and s_1 dominates s_2.
3. No loop nested within L contains a definition of k.
4. k increases when j does: $b_j / b_k > 0$.

Array-Bounds Checking

Objective

Insert test in preheader so that $0 \leq k < n$ in the loop.

Lower Bound

- Let $\Delta k_1, \ldots, \Delta k_m$ be the loop-invariant values added to k inside the loop
- $k \geq 0$ everywhere in the loop if
  - $k \geq 0$ in the loop preheader
  - $\Delta k_1 \geq 0 \ldots \Delta k_m \geq 0$

Array-Bounds Checking

Upper Bound

- Let $\Delta k_1, \ldots, \Delta k_p$ be the set of loop-invariant values added to k on any path between s_1 and s_2 that does not go through s_1.
- $k < n$ at s_2 if $k < n - (\Delta k_1 + \cdots + \Delta k_p)$ at s_1
- From $(k - a_k)/b_k = (j - a_j)/b_j$ this test can be rewritten to $j < b_j/b_k(n - (\Delta k_1 + \cdots + \Delta k_p) - a_k) + a_j$
- It is sufficient that $u \leq b_j/b_k(n - (\Delta k_1 + \cdots + \Delta k_p) - a_k) + a_j$ because the test $j < u$ dominates the test $k < n$
- All parts of this test are loop-invariant!

Array-Bounds Checking Transformation

- Hoist loop-invariants out of the loop
- Copy the loop L to a new loop L' with header label L'_h
- Replace the statement "if $k <_u n$ goto L'_3 else goto L'_4" by "goto L'_3"
- At the end of L's preheader put statements equivalent to
  if $k \geq 0 \wedge \Delta k_1 \geq 0 \wedge \cdots \wedge \Delta k_m \geq 0$
  and $u \leq b_j / b_k (n - (\Delta k_1 + \cdots + \Delta k_p) - a_k) + a_j$
  goto L'_h else goto L_h

Array-Bounds Checking Transformation

- This condition can be evaluated at compile time if
  1. all loop-invariants in the condition are constants; **or**
  2. n and u are the same temporary, $a_k = a_j$, $b_k = b_j$ and no Δk's are added to k between s_1 and s_2.
- The second case arises for instance with code like this:

```
int u = a.length;
int i = 0;
while (i<u) {
  sum += a[i];
  i++;
}
```

  assuming common subexpression elimination for `a.length`
- Compile-time evaluation of the condition means to unconditionally use L or L' and delete the other loop
- Clean up with elimination of unreachable and dead code

Array-Bounds Checking Generalization

- Comparison of $j \leq u'$ instead of $j < u$
- Loop exit test at end of loop body: A test
  - s_2 : if $j < u$ goto L_1 else goto L_2

  where L_2 is out of the loop and s_2 dominates all loop back edges; the Δk_i are between s_2 and any back edge and between the loop header and s_1
- Handle the case $b_j / b_k < 0$
- Handle the case where j counts downward and the loop exit tests for $j \geq l$ (a loop-invariant lower bound)
- The increments to the induction variable may be "undisciplined" with non-obvious increment:

```
while (i<n-1) {
  if (sum<0) { i++; sum += i; i++ } else { i += 2; }
  sum += a[i];
}
```

Outline

Loop Unrolling

- For loops with small body, some time is spent incrementing the loop counter and testing the exit condition
- Loop unrolling optimizes this situation by putting more than one copy of the loop body in the loop
- To unroll a loop L with header h and back edges $s_i \to h$:
  1. Copy L to a new loop L' with header h' and back edges $s_i' \to h'$
  2. Change the back edges in L from $s_i \to h$ to $s_i \to h'$
  3. Change the back edges in L' from $s_i' \to h'$ to $s_i' \to h$

Loop Unrolling Example (Still Useless)

before

$$
\begin{array}{l}
L_1 : \\
\qquad \begin{array}{lcl} x & \leftarrow & M[i] \\ s & \leftarrow & s + x \\ i & \leftarrow & i + 4 \\ \multicolumn{3}{l}{\text{if } i < n \text{ goto } L_1 \text{ else } L_2} \end{array} \\
L_2
\end{array}
$$

after

$$
\begin{array}{l}
L_1 : \\
\qquad \begin{array}{lcl} x & \leftarrow & M[i] \\ s & \leftarrow & s + x \\ i & \leftarrow & i + 4 \\ \multicolumn{3}{l}{\text{if } i < n \text{ goto } L_1' \text{ else } L_2} \end{array} \\
L_1' : \\
\qquad \begin{array}{lcl} x & \leftarrow & M[i] \\ s & \leftarrow & s + x \\ i & \leftarrow & i + 4 \\ \multicolumn{3}{l}{\text{if } i < n \text{ goto } L_1 \text{ else } L_2} \end{array} \\
L_2
\end{array}
$$

Loop Unrolling Improved

- No gain, yet
- Needed: induction variable i such that every increment $i \leftarrow i + \Delta$ dominates every back edge of the loop

$\Rightarrow$ each iteration increments i by the sum of the Δs

$\Rightarrow$ increments and tests can be moved to the back edges of loop

- In general, a separate epilogue is needed to cover the remaining iterations because a loop that is unrolled K times can only do multiple-of-K iterations.

Loop Unrolling Example

$$
\begin{array}{llll}
L_1: & x & \leftarrow & M[i] \\
 & s & \leftarrow & s + x \\
 & x & \leftarrow & M[i+4] \\
 & s & \leftarrow & s + x \\
 & i & \leftarrow & i + 8 \\
 & \multicolumn{3}{l}{\text{if } i < n \text{ goto } L_1 \text{ else } L_2} \\
L_2 & & &
\end{array}
$$

only even numbers

$$
\begin{array}{llll}
 & \multicolumn{3}{l}{\text{if } i < n-4 \text{ goto } L_1 \text{ else } L_2} \\
L_1: & x & \leftarrow & M[i] \\
 & s & \leftarrow & s + x \\
 & x & \leftarrow & M[i+4] \\
 & s & \leftarrow & s + x \\
 & i & \leftarrow & i + 8 \\
 & \multicolumn{3}{l}{\text{if } i < n-4 \text{ goto } L_1 \text{ else } L_2'} \\
L_2': & \multicolumn{3}{l}{\text{if } i < n \text{ goto } L_2 \text{ else } L_3} \\
L_2: & & & \\
 & x & \leftarrow & M[i] \\
 & s & \leftarrow & s + x \\
 & i & \leftarrow & i + 4 \\
L_3 & & &
\end{array}
$$

with epilogue