Compiler Construction 2016/2017
Register Allocation for Programs in SSA-Form

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Outline

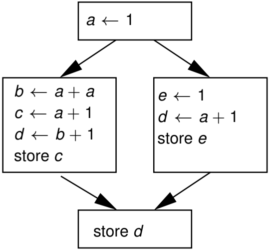
Motivation

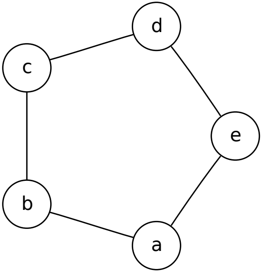
Foundation: Sebastian Hack, Daniel Grund, Gerhard Goos. Towards Register Allocation for Programs in SSA-Form. 2005.

- register allocation maps temporaries to physical registers such that their live ranges do not interfere
- common technique: graph coloring [Chaitin] of the interference graph

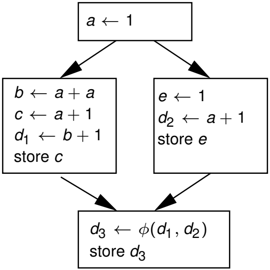
Example: Program and its Interference Graph

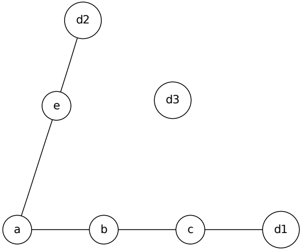
Three Registers Needed





Example Program in SSA Form





- Two registers available: but copy instruction needed
- Three registers available: use all and eliminate copy

SSA and Register Allocation

- ϕ-functions replaced by moves before register allocation
- moves lead to coalescing
- may lead to spill

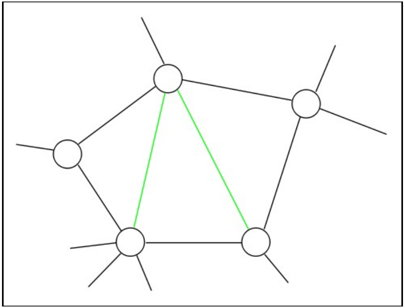
Background

- any undirected graph is inference graph of a program
- finding a minimal k-coloring of a general graph is NP-complete
- hence, the heuristic feedback algorithm
  Build $\rightarrow$ Coalesce $\rightarrow$ Color $\rightarrow$ Spill?
- [coalescing changes colorability of graph]

Background Graph Theory

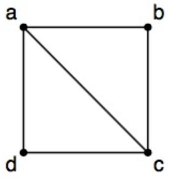
Definition: Chordal Graph, Triangulated Graph

A graph is chordal (also: triangulated) if every cycle of four or more nodes has a chord, i.e., two the nodes from the cycle are connected by an edge that does not belong to the cycle.

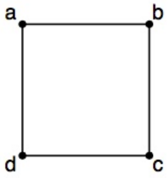


source: http://upload.wikimedia.org/wikipedia/commons/thumb/3/34/Chordal-graph.svg/220px-Chordal-graph.svg.png

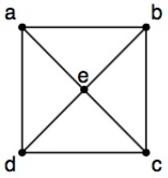
Examples



(a)



(b)



(c)

Background Graph Theory

Definition

- Clique: fully connected subgraph.
- Clique number $\omega(G)$: Size of largest clique of G.
- Chromatic number $\chi(G)$: Minimum k such that G is k-colorable.

Fact

$\omega(G) \leq \chi(G)$

Perfect Graphs

Definition

- If (V, E) is a graph and $V' \subseteq V$,
  then the subgraph induced by V' is $(V', E \cap (V' \times V'))$.
- A graph is perfect if the chromatic number of each induced subgraph is equal to the size of its largest clique.

Facts about perfect graphs

- $\omega(G) = \chi(G)$
- graph coloring can be solved in polynomial time

Graph Coloring and SSA Form

Insight

Interference graphs of SSA programs are chordal graphs
see also [Pereira&Palsberg 2005] [Brisk 2005]
[Bouchez,Darte&Rastello 2005]

Fact

Every chordal graph is perfect

Consequences

- number of registers needed $=$ size of largest clique
  largest $\#$ of variables that are live at the same time
- spilling done once and for all before register allocation
- spilling and coaleascing can be decoupled

Graph Coloring and SSA Form

Direct Path [Pereira&Palsberg 2005]

- A vertex is simplicial if its neighborhood is a clique.
- A simplicial elimination ordering for $G = (V, E)$. is a bijection $\sigma : \{1, \ldots, |V|\} \rightarrow V$ such that $\sigma(i)$ is simplicial in the subgraph induced by $\{\sigma(1), \ldots, \sigma(i)\}$.
- Greedy coloring (i.e., the algorithm that we discussed earlier) is optimal if nodes are selected according to a simplicial elimination ordering.
- Algorithm Maximum Cardinality Search recognizes and determines a simplicial elimination ordering σ of a chordal graph in $O(|E| + |V|)$ time.

Maximum Cardinality Search

MAXIMUMCARDINALITYSEARCH
input: a chordal graph $G = (V, E)$
output: a simplicial elimination ordering σ
for $v \in V$ **do**
$\lambda(v) \leftarrow 0$
for $i \leftarrow 1, \ldots, |V|$ **do**
choose $v \in V$ such that $\forall u \in V$: $\lambda(v) \geq \lambda(u)$
$\sigma(i) \leftarrow v$
for $u \in N(v)$ **do**
$\lambda(u) \leftarrow \lambda(u) + 1$
$V \leftarrow V \setminus \{v\}$

Graph Coloring and SSA Form

But the story does not end here . . .

- Coloring a chordal graph takes $O(|V| + |E|)$ time
- Given the dominator tree and the live ranges, coloring takes $O(\omega(G) \cdot n)$ time
  - n number of instructions
  - $\omega(G)$ size of largest clique in G
    $\leq$ number of registers after spilling
- Usually, ϕ-functions $\mapsto$ move instructions
- Early coalescing is harmful
- Instead of coalescing, try to assign the same color

Outline

ϕ-functions

- ϕ-functions are not functions, but a notational device
- ϕ-functions do not cause interference
- There is no ordering among different ϕ-functions at the beginning of a block; ideally, they should "evaluate" simultaneously

$\Rightarrow$ different notation

$$\begin{array}{ccc} y_1 & \leftarrow & \phi(x_{11}, \ldots, x_{1n}) \\ \vdots & & \vdots \\ y_m & \leftarrow & \phi(x_{m1}, \ldots, x_{mn}) \end{array} \Longrightarrow \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \leftarrow \Phi \begin{bmatrix} x_{11} & \ldots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m_1} & \ldots & x_{mn} \end{bmatrix}$$

Interference Graphs of SSA Programs

Let $\mathcal{D}_v$ be the node defining v and $G = (V, E)$ the interference graph.

Lemma 1

If two registers v and w are live at node n, then either $\mathcal{D}_v$ dominates $\mathcal{D}_w$ or $\mathcal{D}_w$ dominates $\mathcal{D}_v$.

Lemma 2

If v and w interfere and $\mathcal{D}_v$ dominates $\mathcal{D}_w$, then v is live at $\mathcal{D}_w$.

Lemma 3

Let (u, v) and $(v, w) \in E$ be edges, but $(u, w) \notin E$.
If $\mathcal{D}_u$ dominates $\mathcal{D}_v$, then $\mathcal{D}_v$ dominates $\mathcal{D}_w$.

Interference Graphs of SSA Programs are Chordal

Proof of chordality

Consider a cycle of length $n \geq 4$ in the interference graph:

$$x_1 - x_2 - \cdots - x_i - \cdots - x_n - x_1$$

but no edges between x_i and x_j, for $1 \leq i < j < n$ and $j - i > 1$.
Assume that $\mathcal{D}_{x_1} \text{ dom } \mathcal{D}_{x_2}$. By induction, using Lemma 3, $\mathcal{D}_{x_i} \text{ dom } \mathcal{D}_{x_{i+1}}$, for $1 \leq i < n$.
By the edge (x_1, x_n), there is some block ℓ where x_1 and x_n are live and ℓ must be dominated by all $\mathcal{D}_{x_i}$, for $1 \leq i \leq n$. Thus, for each x_i ($i > 1$) there is a path from $\mathcal{D}_{x_i}$ to ℓ, which does not go through $\mathcal{D}_{x_1}$. Hence, the edge (x_1, x_i) must be in the graph.
Contradiction.

Outline

Spilling

- Problem: the interference graph does not reflect the number of uses of a register

$\Rightarrow$ $\exists$ work to break the live ranges in smaller pieces

- Bouchez [2005] shows that “splitting live ranges to lower the register pressure to a fixed k while inserting a minimum number of reload instructions is NP-complete”

A Foundation for Spilling

Lemma

For each clique $C \subseteq G$ with $V_C = \{v_1, \ldots, v_n\}$, there is a permutation $\sigma : V_C \to V_C$ such that $\mathcal{D}_{\sigma(v_i)}$ dominates $\mathcal{D}_{\sigma(v_{i+1})}$ for $1 \leq i < n$.

Theorem

Let G be the interference graph of an SSA program and C be an induced subgraph of G. C is a clique in G iff there exists a label in the program where all V_C are live.

Spilling with Belady's Algorithm

- Let ℓ be a node where $l > k$ variables are live
- Belady's algorithm spills those $l - k$ variables whose uses are farthest away (in minimum number of instructions executed) from ℓ as computed by *nuse*.

$$nuse(\ell, v) = \begin{cases} \infty & \text{if } v \text{ not live at } \ell \\ 0 & \text{if } v \text{ used at } \ell \\ nuse'(\ell, v) & \text{otherwise} \end{cases}$$

$$nuse'(\ell, v) = 1 + \min_{\ell' \in succ[\ell]} nuse(\ell', v)$$

- Apply Belady's algorithm to each basic block B

Belady's Algorithm for Basic Block B

- Let P be the set of variables passed into block B: the variables live-in at B and the results of the ϕ-functions
- Let $\sigma : P \to P$ be a permutation which sorts P ascendingly according to *nuse*

$\Rightarrow$ Pass the set of variables $J = \{p_{\sigma(1)}, \ldots, p_{\sigma(\min(k,l))}\}$ in registers

- Traverse the nodes in a basic block from entry to exit.
- Let Q be the set of all variables currently in registers ($|Q| \leq k$, initially $Q \leftarrow J$)

Belady's Algorithm for Basic Block B

continued

- At instruction $\ell : \underbrace{(y_1, \ldots, y_m)}_{\mathcal{D}_\ell} \leftarrow \tau \underbrace{(x_1, \ldots, x_n)}_{\mathcal{U}_\ell}$

  set $R \leftarrow \mathcal{U}_\ell \setminus Q$
- if $R \neq \emptyset$, then
  - reloads have to be inserted and $\max(|R| + |Q| - k, 0)$ variables are removed from Q
  - remove those with highest *nuse*
- If $v \in J$ is displaced before used, then v need not be passed to B in a register.
- Let in_B be the set $v \in J$ which are used in B before they are displaced.

Belady's Algorithm for Basic Block B

continued

- Instruction τ displaces $\max(|\mathcal{D}_\ell| + |Q| - k, 0)$ variables from Q
- To decide which variables to displace we use $\mathit{nuse}'(\ell, v)$
- Let out_B be the set Q after processing the last node in a block

Belady's Algorithm Extended

- To connect the blocks, ensure that each variable in in_B is in a register on entry to B.
- At the end of each predecessor P' of B insert reloads for all $in_B \setminus out_{P'}$ (recall edge splitting)

Outline

Coloring Chordal Graphs

- perfect elimination orderings (PEO)
- ordering in which variables are removed from graph
- basis: simplicial nodes (all neighbors belong to the same clique)
- **Lemma:** Every chordal graph has a simplicial node.
- Removing a node from a chordal graph preserves chordality
- PEOs are related to the dominance tree

Coloring Chordal Graphs

Theorem

An SSA variable v can be added to a PEO of G if all variables whose definitions are dominated by the definition of v have been added to the PEO.

Proof

For a contradiction, assume v is not simplicial. Hence, v has two neighbors a and b which are not connected.
As all variables whose definitions are dominated by $\mathcal{D}_v$ are already part of the PEO and removed, it must be that $\mathcal{D}_a$ dominates $\mathcal{D}_v$. By a previous lemma, $\mathcal{D}_v$ dominates $\mathcal{D}_b$, contradicting the assumption.

Coloring Chordal Graphs

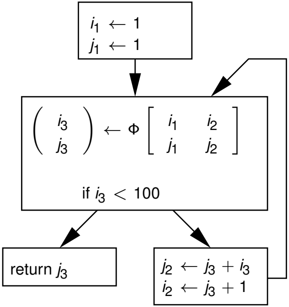
```
ColorProgram (Program P)
    ColorRecursive (entry block of P)

ColorRecursive (Basic block B)
    assigned ← colors of the live-in(B)
    for each instruction (b_1, ..., b_m) ← τ(a_1, ..., a_n) from entry
    to exit do
        for a ∈ {a_1, ..., a_n} do
            if last use of a then
                assigned ← assigned \ color(a)
        for b ∈ {b_1, ..., b_n} do
            color(b) ← one of allcolors \ assigned
    for each C where B = idom(C) do
        ColorRecursive(C)
```

Outline

Coalescing Phase

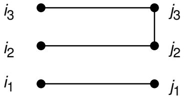
- Goal: minimize number of copy/move instructions
- Causes of copy/move instructions
  - ϕ-functions
  - register constraints of target architecture (pre-colored nodes)

Implementation of ϕ-functions



- Seems to require two registers
- However, implementing Φ by the moves $i_3 \leftarrow i_2; j_3 \leftarrow j_2$ creates an interference between i_3 and j_2

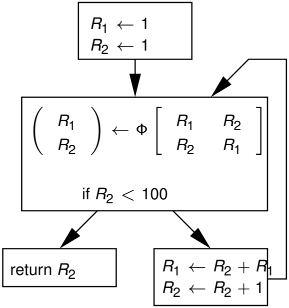
Interference from Implementation of Φ



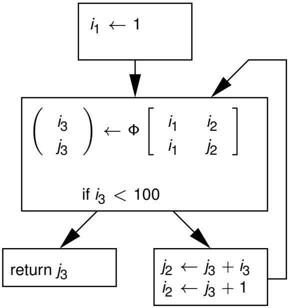
Removal of Φ without Using Extra Registers

- Consider $(b_1, \ldots, b_n) \leftarrow \sigma(a_1, \ldots, a_n)$
- A multi-assignment that permutes the contents of the registers according to σ
- For the example program, a permutation is needed that swaps two registers:

Example Program After Register Assignment

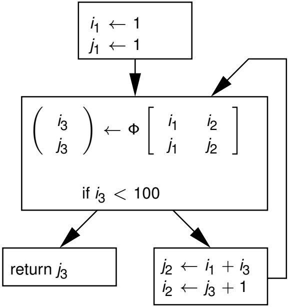


Example Where Copying is Needed



- Φ duplicates i_1 into i_3 and j_3

Example Where Copying is Needed



- i_1 interferes with Φ

Duplication in the Removal of Φ

- Duplication (i.e., extra registers) are only needed if
  - a Φ argument is used multiple times in one column
  - a Φ argument is live-in at the block of Φ
- Interference with a value defined by Φ does not require duplication.

Implementation of Permutations

Register swaps Swap instructions of the processor;
xor trick: $a \leftarrow a \oplus b$; $b \leftarrow a \oplus b$; $a \leftarrow a \oplus b$

Moves assuming a free backup register, each cycle C can be implemented with $|C| + 1$ move instructions
for example, `$at` in MIPS

Optimizing Φ-functions

- The cost of implementation for a permutation σ is related to the number of fixpoints of σ
- Variable x is a fixpoint if

$$(\ldots, x', \ldots) = \sigma(\ldots, x, \ldots)$$

and x and x' are assigned the same register

$\Rightarrow$ no code needs to be generated for a fixpoint

Optimizing Φ-functions

Problem Statement

$$\ell : \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \leftarrow \Phi \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m_1} & \dots & x_{mn} \end{bmatrix}$$

Given a k-coloring $f : V \to \{1, \dots, k\}$ define the cost of ℓ by

$$c_f(\ell) = \sum_{i=1}^{m} \sum_{j=1}^{n} cost_f(y_i, x_{ij})$$

where $cost_f(a, b) = \begin{cases} w_{ab} & \text{if } f(a) \neq f(b) \\ 0 & \text{otherwise} \end{cases}$ with $w_{ab} \geq 0$ the cost of copying b to a.

The overall cost of a program P under coloring f is

$$c(P, f) = \sum_{\ell \text{ is } \Phi\text{-node}} c_f(\ell)$$

SSA-Maximize-Fixed-Points

Given an SSA program P and its interference graph G. Find a coloring f of G for which $c(P, f)$ is minimal.

Theorem

SSA-Maximize-Fixed-Points is NP-complete.

Heuristics for Optimizing Φ-functions

- Start with a *k*-coloring
- Modify color assignments to lower the cost
  Non-local changes in the coloring may be required!
- A valid *k*-coloring is always maintained
- For each row *i* of the Φ-function

$$\begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix} \leftarrow \Phi \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m_1} & \dots & a_{mn} \end{bmatrix}$$

  define an <u>optimization unit</u> (OU) consisting of p_i and all a_{ij} that do not interfere with p_i (at least one)

Perm-Optimizer

```
COALESCE(G)
  pinned ← ∅
  for each OU (p, a_1, ..., a_k) do
    for each color c assignable to p do {Init}
      C_c ← G[p, a_1, ..., a_k] { conflict graph }
      S_c ← max weighted stable subset of C_c {weight of a_i is w_{pa_i} }
      Insert (c, C_c, S_c) in min-queue Q { ordered by w(S_c) }
    repeat { Test }
      candidates ← ∅
      g ← f {copy the current coloring}
      pop (c, C, S) from Q
      C′ ← TEST(c, C, S)
      if C′ ≠ nil then
        S′ ← maximum weighted stable subset of C′
        Insert (c, C′, S′) into Q
    until C′ = nil
    if |candidates| > 1 then
      pinned ← pinned ∪ candidates
      f ← g { update coloring }
```

Perm-Optimizer II

```
TEST(c, C, S)
  { S = {p, a_1, ..., a_l} processed in this order }
  for u ∈ S do
    (s, v) ← TRYCOLOR(u, nil, c)
    if s = ok then
      candidates = candidates ∪ {u}
    else if  s = candidate and v ≠ p then
      return  (V_C, E_C ∪ {(v, u)})
    else
      return  (V_C, E_C ∪ {(u, u)})
  return  nil
```

Perm-Optimizer III

$\textsc{TryColor}(v \in V_G, u \in V_G, c)$

```
    c_v ← g(v)
    if c = c_v then
        return (ok, nil)
    else if v ∈ pinned then
        return (pinned, v)
    else if v ∈ candidates then
        return (candidate, v)
    else if c is not allowed for v then
        return (forbidden, v)
    for each n with (v, n) ∈ E_G, n ≠ u, g(n) = c do
        { try to swap colors with neighbor }
        (s, v′) ← TryColor(n, v, c_v)
        if s ≠ ok then
            return (s, v′)
    g(v) ← c
    return (ok, nil)
```

Outline

Register Constraints

- Most processor architectures have instructions where the operands are restricted to specific registers
- Graph coloring approach
  1. split live range at constraining definition
  2. add one pre-colored node for each register
  3. connect definition with all pre-colored nodes, except the one with the required color
- For chordal graphs, coloring is in P iff each color is used only once in pre-coloring.
  Unrealistic constraint for register allocation

$\Rightarrow$ Delegate to the Perm-Optimizer

Register Constraints by Perm-Optimization

- Insert $(a_i') = \Phi[a_i]$ (for all live registers) in front of each instruction with register constraints

$\Rightarrow$ all live variables can change register at that point

$\Rightarrow$ interference graph breaks in two unconnected components

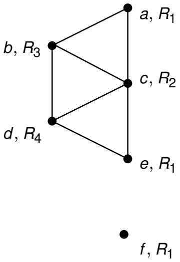
$\Rightarrow$ each color occurs only once as pre-coloring in each component

- first do coloring, then Perm-Optimization

Example Register Constraints

Code and Colored Interference Graph

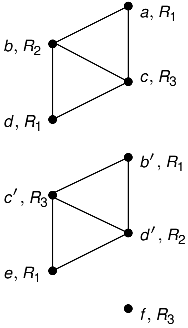
$$
\begin{aligned}
a_{R_1} &\leftarrow \ldots \\
b &\leftarrow \ldots \\
c &\leftarrow b + 1 \\
d &\leftarrow a + 1 \\
e_{R_1} &\leftarrow b + c \\
f &\leftarrow c + d \\
&\vdots
\end{aligned}
$$



Example Register Constraints with Φ Inserted

Code and Colored Interference Graph

$$
\begin{aligned}
a_{R_1} &\leftarrow \ldots \\
b &\leftarrow \ldots \\
c &\leftarrow b + 1 \\
d &\leftarrow a + 1 \\
\begin{pmatrix} b' \\ c' \\ d' \end{pmatrix} &\leftarrow \Phi \begin{bmatrix} b \\ c \\ d \end{bmatrix} \\
e_{R_1} &\leftarrow b' + c' \\
f &\leftarrow c' + d' \\
&\vdots
\end{aligned}
$$



Outline

Conclusion

- Interference graphs for SSA programs are chordal

$\Rightarrow$ main phases of register allocation (spilling, coloring, coalescing) can be decoupled

- Procedure for spilling based on the correspondence live sets $\leftrightarrow$ cliques in interference graph (without constructing the graph)
- (Optimal spilling via ILP solving)
- Optimal coloring in linear time (w/o constructing the graph)
- Optimal coalescing is NP-complete
  - Heuristic
  - (Optimal coalescing via ILP solving)
- Register constraints expressible

Alternatives

- Pereira&Palsberg [APLAS 2005] observe that 95% of the methods in the Java 1.5 library give rise to chordal interference graphs and give an algorithm for register allocation under this assumption
- Pereira&Palsberg [PLDI 2008] give a general, industrial strength framework for register allocation based on puzzle solving. It first transforms its input to elementary programs, a strengthening of SSA programs.
- Pereira&Palsberg [CC 2009] propose a different, spill-free way to perform SSA elimination after register coloring
- Pereira&Palsberg [CC 2010] present Punctual Coalescing, a scalable, linear time, locally optimal algorithm for coalescing.
- Hack&Good [PLDI 2008] register coalescing by graph recoloring.
- Braun&Hack [CC 2009] present an improved spilling algorithm for programs in SSA form.