# Einführung in Agda

https://tinyurl.com/bobkonf17-agda

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## Programs that work — the dependent stairway

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- Express your specification as a type

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- Express your specification as a type
- Write the only possible program of this type

# Experienced Programmer

# Why does it work?

## The Curry-Howard Correspondence

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- Propositions as types
- Proofs as programs

# Why does it work?

## The Curry-Howard Correspondence

- Propositions as types
- Proofs as programs

## Central insight

Write program of this type

=

Find a proof for this proposition

# In Agda

## Remember Curry-Howard

- A type corresponds to a proposition
- **Elements** of the type are **proofs** for that proposition

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A function $f : A \to B$ ...

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A function $f : A \to B$ ...

- transforms an element of $A$ to an element of $B$
- transforms a proof of $A$ to a proof of $B$
- shows: if we have a proof of $A$, then we have a proof of $B$

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## The role of functions

A function $f : A \to B$ ...

- transforms an element of $A$ to an element of $B$
- transforms a proof of $A$ to a proof of $B$
- shows: if we have a proof of $A$, then we have a proof of $B$
- **is** a proof of the logical implication $A \to B$

# Plan

# Logic in Agda
## Defining types: the true proposition

```
-- Truth
data ⊤ : Set where
  tt : ⊤
```

```
-- Truth
data ⊤ : Set where
  tt : ⊤
```

## Explanation (cf. data in Haskell)

- - Truth a comment
- data defines a new datatype
- ⊤ is the name of the type
- Set is its kind
- tt is the single element of ⊤

```
-- Conjunction
data _∧_ (P Q : Set) : Set where
  ⟨_,_⟩ : P → Q → (P ∧ Q)
```

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data _∧_ (P Q : Set) : Set where
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```

## Explanation

- _∧_ the name of an infix type constructor
  the underlines indicate the positions of the arguments
- (P Q : Set) parameters of the type
- ⟨_,_⟩ data constructor with two parameters

# Logic in Agda
## Disjunction is really just `Either`

```
-- Disjunction
data _∨_ (P Q : Set) : Set where
  inl : P → (P ∨ Q)
  inr : Q → (P ∨ Q)
```

```
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data _∨_ (P Q : Set) : Set where
  inl : P → (P ∨ Q)
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```

## Explanation

- two data constructors
- everything covered

# A first program in Agda

## Specification

```
-- Conjunction is commutative
commConj1 : (P : Set) → (Q : Set) → (P ∧ Q) → (Q ∧ P)
```

# A first program in Agda

## Specification

```
-- Conjunction is commutative
```

commConj1 : $(P : \mathsf{Set}) \to (Q : \mathsf{Set}) \to (P \land Q) \to (Q \land P)$

## Explanation

- $(P : \mathsf{Set})$ an argument of type Set with name $P$ to be used later in the type
- $(P : \mathsf{Set})$ and $(Q : \mathsf{Set})$ declare that $P$ and $Q$ are types (propositions)
- $(P \land Q) \to (Q \land P)$ is the proposition we want to prove = the type of the program we want to write

# A first program in Agda

## Specification

```
-- Conjunction is commutative
commConj1 : (P : Set) → (Q : Set) → (P ∧ Q) → (Q ∧ P)
```

## Let's write it interactively

# Should start with a screen like this

```
module LogicGaps where

-- Truth
data ⊤ : Set where
  tt : ⊤

-- Conjunction
data _∧_ (P Q : Set) : Set where
  ⟨_,_⟩ : (p : P) → (q : Q) → (P ∧ Q)

-- Disjunction
data _∨_ (P Q : Set) : Set where
  inl : (p : P) → (P ∨ Q)
  inr : (q : Q) → (P ∨ Q)

-- Conjunction is commutative
commConj1 : (P : Set) → (Q : Set) → P ∧ Q → Q ∧ P
commConj1 = { }0
```

```
?0 : (P Q : Set) → P ∧ Q → Q ∧ P
?1 : {P Q : Set} → P ∨ Q → Q ∨ P
?2 : {P Q R : Set} → (P ∨ Q) ∧ R → (P ∧ R) ∨ (Q ∧ R)
```

# Variations on the specification

## Fully explicit

```
-- Conjunction is commutative
commConj1 : (P : Set) → (Q : Set) → (P ∧ Q) → (Q ∧ P)

commConj1 P Q ⟨ p , q ⟩ = ⟨ q , p ⟩
```

- arguments $P$ and $Q$ are not used and Agda can infer them

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```

- arguments $P$ and $Q$ are not used and Agda can infer them

## With inferred parameters

```
-- Conjunction is commutative
commConj2 : (P Q : Set) → (P ∧ Q) → (Q ∧ P)
commConj2 _ _ ⟨ p , q ⟩ = ⟨ q , p ⟩
```

- just put _ for inferred arguments

# Variations on the specification

## Implicit parameters

```
-- Conjunction is commutative
commConj : ∀ {P Q} → (P ∧ Q) → (Q ∧ P)
commConj ⟨ p , q ⟩ = ⟨ q , p ⟩
```

# Variations on the specification

## Implicit parameters

```
-- Conjunction is commutative
commConj : ∀ {P Q} → (P ∧ Q) → (Q ∧ P)
commConj ⟨ p , q ⟩ = ⟨ q , p ⟩
```

## Explanation

- ∀ {P Q} is short for {P Q : Set}
- {P Q : Set} indicates that $P$ and $Q$ are **implicit parameters**: they need not be provided and Agda tries to infer them
- Successful here, but we get an obscure error message if Agda cannot infer implicit parameters

# A second program in Agda

## Specification

```
-- Disjunction is commutative
commDisj : ∀ {P Q} → (P ∨ Q) → (Q ∨ P)
```

# A second program in Agda

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**Specification**

```
-- Disjunction is commutative
commDisj : ∀ {P Q} → (P ∨ Q) → (Q ∨ P)
```

**Let's write it interactively**

```
-- Falsity
data ⊥ : Set where

-- Negation
¬ : Set → Set
¬ P = P → ⊥
```

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data ⊥ : Set where


-- Negation
¬ : Set → Set
¬ P = P → ⊥
```

## Explanation

- The type ⊥ has **no** elements, hence no constructors
- Negation is defined by *reductio ad absurdum*: $P \rightarrow \bot$
  i.e., having a proof for $P$ would lead to a contradiction

# De Morgan's laws

## Specification

```
-- DeMorgan's laws
demND1 : ∀ {P Q} → ¬ (P ∨ Q) → (¬ P ∧ ¬ Q)
demND2 : ∀ {P Q} → (¬ P ∧ ¬ Q) → ¬ (P ∨ Q)
```

# De Morgan's laws

**Specification**

```
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## Interaction time

# Plan

# Numbers in Agda

## Surprise

- Numbers are not predefined in Agda
- We have to define them ourselves
- (But there is a library)

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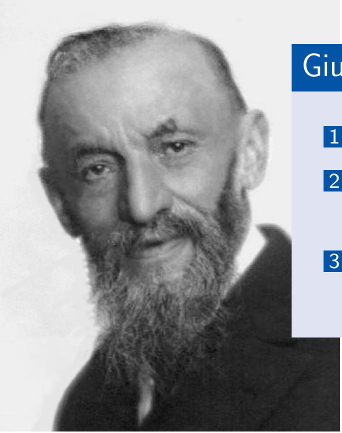
**Let's try**

# Peano's axioms[1]



**Giuseppe Peano says ...**

1. zero is a natural number
2. If $n$ is a natural number, then suc $n$ is also a natural number
3. All natural numbers can be (and must be) constructed from 1. and 2.

[1] Image Attribution: By Unknown - School of Mathematics and Statistics, University of St Andrews, Scotland [1], Public Domain, `https://commons.wikimedia.org/w/index.php?curid=2633677`

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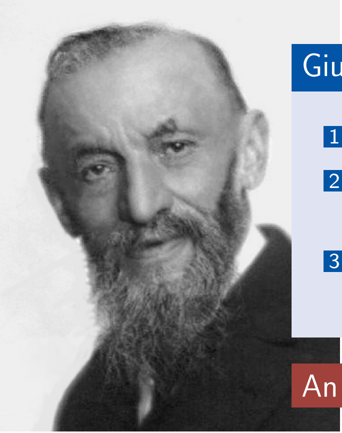

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**An inductive definition**

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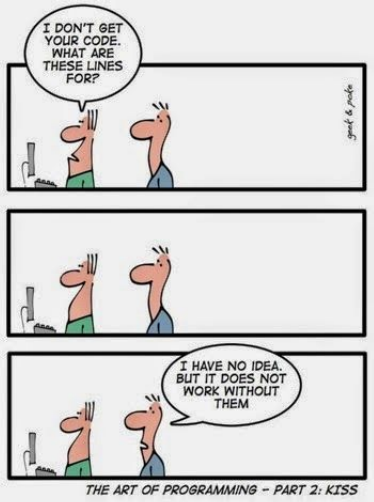
# Inductive definition in Agda

## Natural numbers

```
data ℕ : Set where
  zero : ℕ
  suc : ℕ → ℕ
```



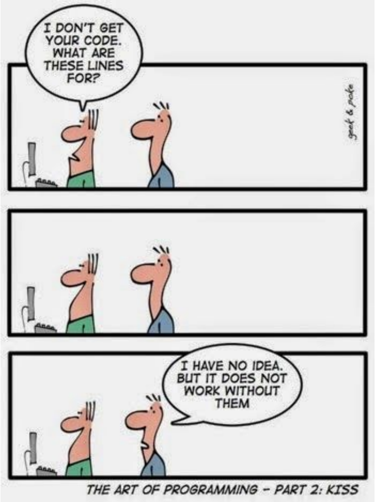
# Inductive definition in Agda

## Natural numbers

```
data ℕ : Set where
  zero : ℕ
  suc : ℕ → ℕ
```

## Explanation

- Defines zero and suc just like demanded by Peano
- Define functions on ℕ by induction and pattern matching on the constructors



# Functional programming

## Addition

$$\begin{aligned}
&\mathsf{add} : \mathbb{N} \to \mathbb{N} \to \mathbb{N}\\
&\mathsf{add}\ \mathsf{zero}\ n = n\\
&\mathsf{add}\ (\mathsf{suc}\ m)\ n = \mathsf{suc}\ (\mathsf{add}\ m\ n)
\end{aligned}$$

# Functional programming

## Addition

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$$\text{add zero } n = n$$
$$\text{add } (\text{suc } m)\ n = \text{suc } (\text{add } m\ n)$$

## Subtraction

$$\text{sub} : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$$
$$\text{sub } m \text{ zero} = m$$
$$\text{sub zero } (\text{suc } n) = \text{zero}$$
$$\text{sub } (\text{suc } m)\ (\text{suc } n) = \text{sub } m\ n$$

# Why specify properties?

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## Deficiency of Testing

Testing shows the presence, not the absence of bugs.

E.W. Dijkstra

# What can we specify?

- Properties of addition all require equality on numbers

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## Next surprise

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**Next surprise**

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# Let's try

# Inductive definition of equality

## Equality on natural numbers

```
data _≡_ : ℕ → ℕ → Set where
  z≡z : zero ≡ zero
  s≡s : {m n : ℕ} → m ≡ n → suc m ≡ suc n
```

## Explanation

- Unusual: datatype parameterized by two numbers
- The constructor s≡s takes a proof that $m \equiv n$ and thus becomes a proof that suc $m \equiv$ suc $n$

# Properties of equality

## Equality is ...

```
-- reflexive
refl-≡ : (n : ℕ) → n ≡ n
-- transitive
trans-≡ : {m n o : ℕ} → m ≡ n → n ≡ o → m ≡ o
-- symmetric
symm-≡ : {m n : ℕ} → m ≡ n → n ≡ m
```

# Properties of equality

## Reflexivity

- Need to define a function that given some $n$ returns a proof of (element of) $n \equiv n$
- Straightforward programming exercise
- Use pattern matching / induction
- Agda can do it automatically

# Properties of equality

## Reflexivity

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# Interaction time

# Properties of equality

### Symmetry

- $m \equiv n \rightarrow n \equiv m$
- Symmetry can be proved by induction on $m$ and $n$
- Introduces a new concept: absurd patterns
- Less cumbersome alternative:
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# Properties of equality

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# Properties of addition

## Zero is neutral element of addition

$$\text{neutralAdd0l} : (m : \mathbb{N}) \rightarrow \text{add zero } m \equiv m$$
$$\text{neutralAdd0r} : (m : \mathbb{N}) \rightarrow \text{add } m \text{ zero} \equiv m$$

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## Addition is associative

$$\text{assocAdd} : (m\ n\ o : \mathbb{N}) \to \text{add } m\ (\text{add } n\ o) \equiv \text{add } (\text{add } m\ n)\ o$$

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## Addition is commutative

$$\text{commAdd} : (m\ n : \mathbb{N}) \to \text{add } m\ n \equiv \text{add } n\ m$$

# Properties of addition

## Proving ...

- Neutral element and associativity are straightforward
- Commutativity is slightly more involved
- Requires an auxiliary function

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## Interaction time

# Plan

# Vectors in Agda

## Vectors with static bounds checks

- Flagship application of dependent typing
- All vector operations proved safe at compile time
- Key: define vector type indexed by its length

# The vector type

```
data Vec (A : Set) : (n : ℕ) → Set where
  Nil : Vec A zero
  Cons : {n : ℕ} → (a : A) → Vec A n → Vec A (suc n)
```

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```

```
concat : ∀ {A m n}
       → Vec A m → Vec A n → Vec A (add m n)
concat Nil ys = ys
concat (Cons a xs) ys = Cons a (concat xs ys)
```

# Safe vector access
"avoid out of bound indexes"

## Trick #1

- Type of get depends on length of vector $n$ and index $m$
- … and a proof that $m < n$

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$$\text{get} : \forall \{A\ n\} \to \text{Vec}\ A\ n \to (m : \mathbb{N}) \to \text{suc}\ m \leq n \to A$$

## Safe vector access
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### Trick #2

- ... type restricts the index to $m < n$

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## Trick #2

- ... type restricts the index to $m < n$

$$\text{get1} : \forall \{A\ n\} \to \text{Vec}\ A\ n \to \text{Fin}\ n \to A$$

# Finite set type

```
data Fin : ℕ → Set where
  zero : {n : ℕ} → Fin (suc n)
  suc : {n : ℕ} → Fin n → Fin (suc n)
```

# Finite set type

```
data Fin : ℕ → Set where
  zero : {n : ℕ} → Fin (suc n)
  suc : {n : ℕ} → Fin n → Fin (suc n)
```

## Explanation

- Overloading of constructors ok
- Fin zero $= \emptyset$ (empty set)
- Fin (suc zero) $= \{0\}$
- Fin (suc (suc zero)) $= \{0, 1\}$
- etc

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- etc

# Interaction time

# Splitting a vector

## We know this type already . . .

```
-- Pair
data _×_ (A B : Set) : Set where
  _,_ : (a : A) → (b : B) → (A × B)
```

```
-- split a vector in two parts
split : ∀ {A n} → Vec A n → (m : ℕ) → m ≤ n
      → Vec A m × Vec A (sub n m)
```

- Solution introduces a new feature: with matching
- This operation can also be defined with Fin . . .

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- This operation can also be defined with Fin . . .

# Plan

# Going further

- http://learnyouanagda.liamoc.net/ nicely paced tutorial, some more background
- http://wiki.portal.chalmers.se/agda/pmwiki.php?n=Main.HomePage definitive resource
- http://wiki.portal.chalmers.se/agda/pmwiki.php?n=Main.Othertutorials with a load of links to tutorials

# Questions?