Lecture: Program analysis
Exercise 3
http://proglang.informatik.uni-freiburg.de/teaching/programanalysis/2010ss/

Definitions

1. A complete partial order $(M, \leq)$ has a *flat* ordering iff
$$\forall x, y \in M : x \leq y \Rightarrow x = \bot$$
2. A complete lattice $(M, \leq)$ has a *flat* ordering iff
$$\forall x, y \in M : x \leq y \Rightarrow x = \bot \vee x = y$$
3. Let $(M, \leq)$ and $(N, \leq)$ be complete partial orders, and $f : M \to N$. f is
   (a) *monotone* iff $x \leq y \Rightarrow f(x) \leq f(y)$;
   (b) *strict* iff $f(\bot) = \bot$.
4. Let $(M, \leq)$ and $(N, \leq)$ be complete lattices, and $f : M \to N$. f is *continuous* iff f preserves the least upper bound, i.e. for all chains it holds that
$$f\left(\bigsqcup_{i \in I} x^{(i)}\right) = \bigsqcup_{i \in I} f(x^{(i)})$$

Exercise 1

Given functions $f : M \to N$ and $g : N \to P$, which of the following statements are true? Give a proof or a counter example.

For complete partial orders $(M, \leq)$ and $(N, \leq)$:

1. If $(N, \leq)$ has a flat ordering and f is monotone, then f is strict or constant.
2. If $(M, \leq)$ has a flat ordering and f is strict, then f is monotone.

For complete lattices $(M, \leq)$, $(N, \leq)$,and $(P, \leq)$:

1. If in $(M, \leq)$ every chain is stationary and f is monotone, then f is continuous.
2. If f is monotone, then f is strict.
3. If f and g are monotone (continuous, strict), then $f \circ g$ is monotone (continuous, strict).
4. If f is monotone and $\langle x^{(i)} \rangle_{i \in I}$ is a chain in M, then $\bigsqcup_{i \in I} f(x^{(i)}) \leq f(\bigsqcup_{i \in I} x^{(i)})$.
5. If f is continuous, then it is also monotone.

Definition

Let $(M, \leq)$ be a complete lattice, and $P : M \to \mathbb{B} = \{\texttt{true}, \texttt{false}\}$ a predicate. P is *continuous* iff for every chain $\langle x^{(i)} \rangle_{i \in I}$ in M it holds that $P(x^{(i)}) = \texttt{true}$ for all $i \in I$ implies $P(\bigsqcup_{i \in I} x^{(i)}) = \texttt{true}$.

Exercise 2

Let $(M, \leq)$ be a complete lattice, $f : M \to M$ a continuous function, and $P : M \to \mathbb{B}$ a continuous predicate. Prove that
$$P(\bot) = \texttt{true} \wedge \forall x \in M : (P(x) = \texttt{true} \Rightarrow P(f(x)) = \texttt{true})$$
implies
$$P(\mathit{lfp}(f)) = \texttt{true}$$
where $\mathit{lfp}(f)$ is the smallest fixed point of f.

Exercise 3

Let $(A, \leq)$ and $(G, \leq)$ be partial orders, and (α, γ) be a Galois connection between A and G, i.e. for $X \in G$ and $Y \in A$ it holds:

$$X \leq \gamma(Y) \quad \Longleftrightarrow \quad \alpha(X) \leq Y$$

Which of the following statements are true? Give a proof or a counter example.

1. α monotone
2. γ monotone
3. $\alpha = \alpha \circ \gamma \circ \alpha$
4. $\gamma = \gamma \circ \alpha \circ \gamma$

Exercise 4

Let $(L, \leq)$ be a complete lattice, and $f : L \to L$ a monotone function. If $(L, \leq)$ satisfies the ascending chain condition (ACC), then

$$\mathit{lfp}(f) = \bigsqcup_n f^{(n)}(\bot)$$

Submission

- Deadline: 18.05.2010, 09:00, per mail to `bieniusa@informatik.uni-freiburg.de`, or on paper to Annette Bieniusa, Geb. 079, Room 000-14.
- Late submissions will not be marked.
- Do not forget to put your name on the exercise sheet.
- You might want to read up in Appendix A of *Principles of Program Analysis*.