Lecture: Program analysis
Exercise 3
http://proglang.informatik.uni-freiburg.de/teaching/programanalysis/2010ss/

Definitions

1. A complete partial order $(M, \leq)$ has a *flat* ordering iff

$$\forall x, y \in M : x \leq y \Rightarrow x = \bot \vee x = y$$

2. Let $(M, \leq)$ and $(N, \leq)$ be complete partial orders, and $f : M \to N$. f is
   (a) *monotone* iff $x \leq y \Rightarrow f(x) \leq f(y)$;
   (b) *strict* iff $f(\bot) = \bot$.

3. Let $(M, \leq)$ and $(N, \leq)$ be complete lattices, and $f : M \to N$. f is *(Scott) continuous* iff f preserves least upper bounds of chains, i.e. for all chains it holds that

$$f\left(\bigsqcup_{i \in I} x^{(i)}\right) = \bigsqcup_{i \in I} f(x^{(i)})$$

Exercise 1

Given functions $f : M \to N$ and $g : N \to P$, which of the following statements are true? Give a proof or a counter example.

For complete partial orders $(M, \leq)$ and $(N, \leq)$:

1. If $(N, \leq)$ has a flat ordering and f is monotone, then f is strict or constant.
2. If $(M, \leq)$ has a flat ordering and f is strict, then f is monotone.

For complete lattices $(M, \leq), (N, \leq)$, and $(P, \leq)$:

1. If $(M, \leq)$ satisfies the Ascending Chain Condition and f is monotone, then f is continuous.
2. If f is monotone, then f is strict.
3. If f and g are monotone (continuous, strict), then $g \circ f$ is monotone (continuous, strict).
4. If f is monotone and $\langle x^{(i)} \rangle_{i \in I}$ is a chain in M, then $\bigsqcup_{i \in I} f(x^{(i)}) \leq f(\bigsqcup_{i \in I} x^{(i)})$.
5. If f is continuous, then f is also monotone.

Solution

1. $\forall x \in M : f$ monotone and $\bot \leq x \Rightarrow f(\bot) \leq f(x)$. Since N has a flat ordering, it follows that $f(\bot) = \bot \ \vee f(\bot) = f(x)$. This means that f is either strict ($f(\bot) = \bot$), or f is constant, because for every $x \in M : f(x) = f(\bot)$.

2. Let $x, y \in M$. Since M has a flat ordering, it holds that

$$x \leq y \Rightarrow x = \bot \vee x = y \tag{1}$$

   As f is strict, it follows that

$$f(x) = f(\bot) = \bot \leq f(y) \quad \vee \quad f(x) = f(y) \tag{2}$$

   Therefore $f(x) \leq f(y)$, and f is monotone.

1. Let $\langle x^{(i)} \rangle_{i \in I}$ be an (arbitrary) chain in M. Construct an ascending chain $\langle y^{(j)} \rangle_{j \in \mathbb{N}}$ like this: Take $y^{(0)} = x^{(i)}$ for a $x^{(i)} \in \langle x^{(i)} \rangle_{i \in I}$. Then

$$y^{(j+1)} = \begin{cases} x^{(i)} & \text{such that } \bigsqcup_{k=0}^{j} y^{(k)} \leq x^{(i)} \\ y^{(j)} & \text{otherwise} \end{cases}$$

$\Rightarrow^{ACC} \exists j_0 : y^{(j_0)} = y^{(j_0+1)}$. Hence, $y^{(j_0)} = \bigsqcup_{j \in \mathbb{N}} y^{(j)} = \bigsqcup_{i \in I} x^{(i)}$.

Since f is monotone: $f(y^{(0)}) \leq \cdots \leq f(y^{(j_0)}) = \bigsqcup_{j \in \mathbb{N}} f(y^{(j)})$, and also,

$$\bigsqcup_{j \in \mathbb{N}} f(y^{(j)}) = \bigsqcup_{i \in I} f(x^{(i)}).$$

2. Define partial orders $M = N = (\{\perp, b\}, \leq)$ with $\perp \leq b$, and $f(\perp) = f(b) = b$. Then f is monotone, but not strict.

3.
   - Let $x, y \in M, x \leq y \Rightarrow f(x) \leq f(y) \Rightarrow g(f(x)) \leq g(f(y))$, as f and g are monotone. Hence, $g \circ f$ is monotone.
   - Let $\langle x^{(i)} \rangle_{i \in I}$ be a chain in M.

     $$g\left(f\left(\bigsqcup_{i \in I} x^{(i)}\right)\right) = g\left(\bigsqcup_{i \in I} f\left(x^{(i)}\right)\right) = \bigsqcup_{i \in I} g\left(f\left(x^{(i)}\right)\right)$$

     Hence, $g \circ f$ is continuous.
   - Let $\perp_M \in M$. Then, $f(\perp_M) = \perp_N$ and $g(f(\perp_M)) = g(\perp_N) = \perp_P$. Hence $g \circ f$ is strict.

4. It holds that $x^{(j)} \leq \bigsqcup_{i \in I} x^{(i)}$ for all $j \in I$, and because f is monotone, it follows that

$$f(x^{(j)}) \leq f(\bigsqcup_{i \in I} x^{(i)}) \quad \forall j \in I. \tag{3}$$

Hence, $f(\bigsqcup_{i \in I} x^{(i)})$ is an upper bound for the chain $\langle f(x^{(i)}) \rangle_{i \in I}$, and by definition

$$\bigsqcup_{i \in I} f(x^{(i)}) \leq f(\bigsqcup_{i \in I} x^{(i)}).$$

5. Let $x, y \in M$ with $x \leq y$. Then, $x \sqcup y = y$. Since f is continuous, it follows that

$$f(y) = f(x \sqcup y) = f(x) \sqcup f(y),$$

and hence $f(x) \leq f(y)$.

Definition

Let $(M, \leq)$ be a complete lattice, and $P : M \to \mathbb{B} = \{\texttt{true}, \texttt{false}\}$ a predicate. P is *continuous* iff for every chain $\langle x^{(i)} \rangle_{i \in I}$ in M it holds that $P(x^{(i)}) = \texttt{true}$ for all $i \in I$ implies $P(\bigsqcup_{i \in I} x^{(i)}) = \texttt{true}$.

Exercise 2

Let $(M, \leq)$ be a complete lattice, $f : M \to M$ a continuous function, and $P : M \to \mathbb{B}$ a continuous predicate. Prove that

$$P(\perp) = \texttt{true} \wedge \forall x \in M : (P(x) = \texttt{true} \Rightarrow P(f(x)) = \texttt{true})$$

implies

$$P(\mathit{lfp}(f)) = \texttt{true}$$

where $\mathit{lfp}(f)$ is the smallest fixed point of f.

Solution

By induction, $P(f^i(\bot)) = \texttt{true}$ for all elements in the chain $\bot \leq f(\bot) \leq \ldots$: The base case is $P(\bot) = \texttt{true}$, and the induction step is

$$P(f^i(\bot)) = \texttt{true} \Rightarrow P(f(f^i(\bot)) = \texttt{true} = P(f^{i+1}(\bot)) \tag{4}$$

P is continuous, this means that for every chain $\langle x^{(i)} \rangle_{i \in I}$ in M it holds that $P(x^{(i)}) = \texttt{true}$ for all $i \in I$ implies $P(\bigsqcup_{i \in I} x^{(i)}) = \texttt{true}$. This gives $P(\bigsqcup_{i \geq 0} f^i(\bot)) = \texttt{true}$. The fixed point theorem then gives $\bigsqcup_{i \geq 0} f^i(\bot) = lfp(f)$.

Exercise 3

Let $(A, \leq)$ and $(G, \leq)$ be partial orders, and (α, γ) be a Galois connection between A and G, i.e. for $X \in G$ and $Y \in A$ it holds:

$$X \leq \gamma(Y) \quad \Longleftrightarrow \quad \alpha(X) \leq Y$$

Which of the following statements are true? Give a proof or a counter example.

1. α monotone
2. γ monotone
3. $\alpha = \alpha \circ \gamma \circ \alpha$
4. $\gamma = \gamma \circ \alpha \circ \gamma$

Solution

$\alpha(X) \leq \alpha(X)$ implies $X \leq \gamma(\alpha(X))$, and $\gamma(Y) \leq \gamma(Y)$ implies $\alpha(\gamma(Y)) \leq Y$.

1. $X_1 \leq X_2 \quad \Rightarrow \quad X_1 \leq X_2 \leq \gamma(\alpha(X_2)) \quad \Rightarrow \quad \alpha(X_1) \leq \alpha(X_2)$.
2. $Y_1 \leq Y_2 \quad \Rightarrow \quad \alpha(\gamma(Y_1) \leq Y_1 \leq Y_2 \quad \Rightarrow \quad \gamma(Y_1) \leq \gamma(Y_2)$.
3. It holds that $\alpha(\gamma(\alpha(X))) \leq \alpha(X)$ and $X \leq \gamma(\alpha(\gamma(\alpha(X))))$. Therefore, $\alpha(X) \leq \alpha(\gamma(\alpha(X)))$, and we have shown that $\alpha = \alpha \circ \gamma \circ \alpha$.
4. It holds that $\gamma(Y) \leq \gamma(\alpha(\gamma(Y)))$ and $\alpha(\gamma(\alpha(\gamma(Y)))) \leq Y$. Hence, $\gamma(\alpha(\gamma(Y))) \leq \gamma(Y)$. And finally, $\gamma = \gamma \circ \alpha \circ \gamma$.

Exercise 4

Let $(L, \leq)$ be a complete lattice, and $f : L \to L$ a monotone function. If $(L, \leq)$ satisfies the ascending chain condition (ACC), then

$$\mathit{lfp}(f) = \bigsqcup_{n \in \mathbb{N}} f^{(n)}(\bot)$$

Solution

$\langle f^{(n)}(\bot) \rangle_{n \in \mathbb{N}}$ is an ascending chain: By definition, $\bot \leq f(\bot)$, and monotonicity of f yields $f^{(i)}(\bot) \leq f^{(i+1)}(\bot)$ for all $i \in \mathbb{N}$. By ACC, there exists $n \in \mathbb{N} : f^{(n)}(\bot) = f^{(n+1)}(\bot)$. Hence, $f^{(n)}(\bot) := l_0$ is a fixed point.

Let l be another fixed point, i.e. $l = f(l)$. As $\bot \leq l$ and by monotonicity of f, it holds that

$$f^{(i)}(\bot) \leq f^{(i)}(l) = l \quad \forall i \in \mathbb{N}.$$

Therefore, $l_0 \leq l$, and l_0 is lfp.