Static Program Analysis

http://proglang.informatik.uni-freiburg.de/teaching/programanalysis/2014ss/

Exercise Sheet 7

03.07.2014

Definitions

$t ::=$	*terms :*
x	*variable*
$\lambda x.t$	*abstraction*
$t\ t$	*application*

Figure 1: Syntactic forms of the lambda calculus

1. Let $\mathcal{V}$ be a countable set of variable names. The set of terms is the smallest set $\mathcal{T}$ such that
   a) $x \in \mathcal{T}$ for every $x \in \mathcal{V}$
   b) if $\mathtt{t}_1 \in \mathcal{T}$ and $x \in \mathcal{V}$, then $\lambda.\mathtt{t}_1 \in \mathcal{T}$;
   c) if $\mathtt{t}_1 \in \mathcal{T}$ and $\mathtt{t}_2 \in \mathcal{T}$, then $\mathtt{t}_1\ \mathtt{t}_2 \in \mathcal{T}$;
2. The *size* of a term is defined as

$$\begin{aligned} \mathrm{size}(x) &= 1 \\ \mathrm{size}(\lambda.\mathtt{t}_1) &= \mathrm{size}(\mathtt{t}_1) + 1 \\ \mathrm{size}(\mathtt{t}_1\ \mathtt{t}_2) &= \mathrm{size}(\mathtt{t}_1) + \mathrm{size}(\mathtt{t}_2) + 1 \end{aligned}$$

3. The set of *free variables* of a term $\mathtt{t}$, written $\mathrm{FV}(\mathtt{t})$, is defined inductively as follows:

$$\begin{aligned} \mathrm{FV}(x) &= x \\ \mathrm{FV}(\lambda x.\mathtt{t}_1) &= \mathrm{FV}(\mathtt{t}_1) \setminus x \\ \mathrm{FV}(\mathtt{t}_1\ \mathtt{t}_2) &= \mathrm{FV}(\mathtt{t}_1) \cup \mathrm{FV}(\mathtt{t}_2) \end{aligned}$$

4. The set of *bound variables* of a term $\mathtt{t}$, written $\mathrm{BV}(\mathtt{t})$, is defined inductively as follows:

$$\begin{aligned} \mathrm{BV}(x) &= \emptyset \\ \mathrm{BV}(\lambda x.\mathtt{t}_1) &= x \cup \mathrm{BV}(\mathtt{t}_1) \\ \mathrm{BV}(\mathtt{t}_1\ \mathtt{t}_2) &= \mathrm{BV}(\mathtt{t}_1) \cup \mathrm{BV}(\mathtt{t}_2) \end{aligned}$$

Exercise 1 (Properties of FV)

1. Give a proof that $|\,\mathrm{FV}(\mathtt{t})| \leq \mathrm{size}(\mathtt{t})$ for every term $\mathtt{t}$.

2. Provide an example for a term `t` such that $\mathrm{FV}(\mathtt{t}) \cap \mathrm{BV}(\mathtt{t}) \neq \emptyset$.

Exercise 2 (Equality on traces)
We are now looking at a universe $\mathcal{U} = \mathbf{Trace} \times \mathbf{Trace}$, where $\mathbf{Trace} = \Sigma^*$ is just the set of all finite traces over the alphabet $\Sigma = (\mathbf{Var} \times \mathbf{Lab})$. Let EQ be the equality relation on Σ^*:

$$EQ = \{(v, v) \mid v \in \Sigma^*\}$$

Given the monotone function $F : \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathcal{U})$:

$$F(R) = \{(\epsilon, \epsilon)\} \cup \{(av, aw) \mid a \in \Sigma \text{ and } (v, w) \in R\}$$

- What is $\mathrm{gfp}\, F$?
- Prove equality is the least fixpoint of F:

$$\mathrm{lfp}\, F \stackrel{?}{=} EQ$$

Hint: Consider the definitions of F-consistent (post-fixpoint), F-closed (pre-fixpoint), and the Knaster-Tarski-Theorem. In particular, you can use the principle of induction: if X is F-closed, then $\mathrm{lfp}\, F \subseteq X$. *You can also use Lemma 1.*

Lemma 1

$$\forall j \in \mathbb{N} : F^{(j)}(\emptyset) \subseteq \mathrm{lfp}\, F.$$

Submission In PDF format via email to `geffken AT informatik.uni-freiburg.de`. Please name your single file with the scheme: `ex07-name.pdf`.

- Deadline: **10.07.2014, 12:00**
- Late submissions will not be marked.
- Do not forget to write your name on the exercise sheet.