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Static Program Analysis

http://proglang.informatik.uni-freiburg.de/teaching/programanalysis/2014ss/

Solution Sheet 4

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Exercise 1 (Complete lattices)

1. Let $M = \{a, b, c\}$. Define a relation R such that (M, R) is a complete lattice.
2. For a totally ordered set S, $(\mathcal{P}(S), \subseteq)$ is a complete lattice. Define another relation R such that $(\mathcal{P}(S), R)$ is a complete lattice.
3. Is $(\mathbb{R}, \leq)$ a complete lattice? If not, how can you extend $\mathbb{R}$ such that it becomes a complete lattice?
4. In Exercise 1 on Exercise Sheet 1 you have used the powerset lattice $L_{\mathcal{P}} = (\mathcal{P}(\mathbf{Var}_\star \times \{-, 0, +\}), \subseteq)$ in the Detection of Signs Analysis. Alternatively, you could have used the lattice $L = (\mathbf{Var}_* \to \mathcal{P}(\{-, +, 0\}), \sqsubseteq)$.
   - Provide the definition of the appropriate relation $\sqsubseteq$.
   - What are the values of $\top$ and $\perp$ in L?
   - What is $l_1 \sqcup l_2$?
   - Is there any difference in the precision between the two approaches?

Solution

1. Define $a < b$ and $b < c$. Then $(M, \leq)$ is a complete lattice where $\leq$ is the reflexive and transitive hull of $<$.
2. An easy solution is $(\mathcal{P}(S), \supseteq)$. Another relation can be constructed like this: Let $<$ be a total order on S. You can order all elements of a subset of S with $<$. Using this, you can construct a monotonic sequence of the subset (**Careful!** This does not work in general! For this to work, the subset must have a least element). If we take the lexicographical order $<_l$ on these sequences, we get again a complete lattice for $(\mathcal{P}(S), <_l)$.
3. $(\mathbb{R}, \leq)$ is not a complete lattice. For example, $\sqcup\mathbb{N}$ does not exist. The extension $(\mathbb{R} \cup \{\pm\infty\}, \leq)$ with
   $$\forall x \in \mathbb{R} : -\infty < x < +\infty \tag{1}$$
   is a complete lattice.
4.
   - $\sqsubseteq$ is the pointwise ordering
   - $\top = x \mapsto \emptyset$ and $\perp = x \mapsto \{-, +, 0\}$
   - $(l_1 \sqcup l_2)(x) = l_1(x) \cup l_2(x)$
   - The two lattices are order isomorphic (i.e., there is a bijective map $f : L_{\mathcal{P}} \mapsto L$ defined as $f(x) = \lambda y.\{w \mid (v, w) \in x\}$ such that $x \subseteq y \Leftrightarrow f(x) \sqsubseteq f(y)$).

Definitions

1. A complete partial order $(M, \leq)$ has a *flat* ordering iff
   $$\forall x, y \in M : x \leq y \Rightarrow x = \perp \vee x = y$$

2. Let $(M, \leq)$ and $(N, \leq)$ be pointed complete partial orders, and $f : M \to N$. f is
   a) *monotone* iff $x \leq y \Rightarrow f(x) \leq f(y)$;
   b) *strict* iff $f(\perp) = \perp$.

3. Let $(M, \leq)$ and $(N, \leq)$ be complete lattices, and $f : M \to N$. f is *(Scott) continuous* iff f preserves least upper bounds of chains, i.e. for all chains it holds that

$$f\left(\bigsqcup_{i \in I} x^{(i)}\right) = \bigsqcup_{i \in I} f(x^{(i)})$$

Exercise 2

Given functions $f : M \to N$ and $g : N \to P$, which of the following statements are true? Give a proof or a counter example.

For pointed complete partial orders $(M, \leq)$ and $(N, \leq)$:

1. If $(N, \leq)$ has a flat ordering and f is monotone, then f is strict or constant.
2. If $(M, \leq)$ has a flat ordering and f is strict, then f is monotone.

For complete lattices $(M, \leq), (N, \leq)$, and $(P, \leq)$:

3. If $(M, \leq)$ satisfies the Ascending Chain Condition and f is monotone, then f is continuous.
4. If f is monotone, then f is strict.
5. If f and g are monotone (continuous, strict), then $g \circ f$ is monotone (continuous, strict).
6. If f is monotone and $\langle x^{(i)} \rangle_{i \in I}$ is a chain in M, then $\bigsqcup_{i \in I} f(x^{(i)}) \leq f(\bigsqcup_{i \in I} x^{(i)})$.
7. If f is continuous, then f is also monotone.

Solution

1. $\forall x \in M : f$ monotone and $\perp \leq x \Rightarrow f(\perp) \leq f(x)$. Since N has a flat ordering, it follows that $f(\perp) = \perp \;\vee\; f(\perp) = f(x)$. This means that f is either strict ($f(\perp) = \perp$), or f is constant, because for every $x \in M : f(x) = f(\perp)$.

2. Let $x, y \in M$. Since M has a flat ordering, it holds that

$$x \leq y \Rightarrow x = \perp \vee x = y \tag{2}$$

   As f is strict, it follows that

$$f(x) = f(\perp) = \perp \leq f(y) \quad \vee \quad f(x) = f(y) \tag{3}$$

   Therefore $f(x) \leq f(y)$, and f is monotone.

3. Let $\langle x^{(i)} \rangle_{i \in I}$ be an (arbitrary) chain in M and $x^{(i_0)} = \bigsqcup_{i \in I} x^{(i)}$. Because f is monotone

$$\begin{aligned} &\forall i \in I : f(x^{(i)}) \leq f(x^{(i_o)}) \\ \Longrightarrow\; &f(x^{(0)}) \leq \ldots \leq f(x^{(i_0)}) = \bigsqcup_{i \in \mathbb{I}} f(x^{(i)}) \end{aligned}$$

   It follows that

$$f(\bigsqcup_{i \in I} x^{(i)}) = f(x^{(i_0)}) = \bigsqcup_{i \in I} f(x^{(i)}).$$

4. Define partial orders $M = N = (\{\perp, b\}, \leq)$ with $\perp \leq b$, and $f(\perp) = f(b) = b$. Then f is monotone, but not strict.

5. - Let $x, y \in M, x \leq y \Rightarrow f(x) \leq f(y) \Rightarrow g(f(x)) \leq g(f(y))$, as f and g are monotone. Hence, $g \circ f$ is monotone.
   - Let $\langle x^{(i)} \rangle_{i \in I}$ be a chain in M.

   $$g\left(f\left(\bigsqcup_{i \in I} x^{(i)}\right)\right) = g\left(\bigsqcup_{i \in I} f\left(x^{(i)}\right)\right) = \bigsqcup_{i \in I} g\left(f\left(x^{(i)}\right)\right)$$

     Hence, $g \circ f$ is continuous.
   - Let $\perp_M \in M$. Then, $f(\perp_M) = \perp_N$ and $g(f(\perp_M)) = g(\perp_N) = \perp_P$. Hence $g \circ f$ is strict.

6. It holds that $x^{(j)} \leq \bigsqcup_{i \in I} x^{(i)}$ for all $j \in I$, and because f is monotone, it follows that

   $$f(x^{(j)}) \leq f(\bigsqcup_{i \in I} x^{(i)}) \quad \forall j \in I. \tag{4}$$

   Hence, $f(\bigsqcup_{i \in I} x^{(i)})$ is an upper bound for the chain $\langle f(x^{(i)}) \rangle_{i \in I}$, and by definition

   $$\bigsqcup_{i \in I} f(x^{(i)}) \leq f(\bigsqcup_{i \in I} x^{(i)}).$$

7. Let $x, y \in M$ with $x \leq y$. Then, $x \sqcup y = y$. Since f is continuous, it follows that

   $$f(y) = f(x \sqcup y) = f(x) \sqcup f(y),$$

   and hence $f(x) \leq f(y)$.