# Softwaretechnik

## Program verification

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# Road Map

- Program verification
- Automatic program verification
  - Programs with loops
  - Programs with recursive function calls

# Proving Program Correctness: General Approach

## Program annotation

- Annotation @$F$ at program location $L$ asserts that formula $F$ is true whenever program control reaches $L$
- Special annotation: function specification
  - Precondition = specifies what should be true upon entering
  - Postcondition = specifies what must hold after executing

## Proving Program Correctness

- Input: Program with annotations
- Translate input to first order formula $f$
- Validity of $f$ implies program correctness

# Outline

- Proving partial correctness
  - Programs with loops

# Proving Partial Correctness

## Recall

A function $f$ is partially correct if
when $f$'s precondition is satisfied on entry and $f$ terminates, then $f$'s postcondition is satisfied.

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A function $f$ is partially correct if
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then $f$'s postcondition is satisfied.

## Automatic Verification

- Function + annotation is transformed to finite set of FOL formulae, the verification conditions (VCs)
- If all VCs are valid, then the function obeys its specification (partially correct)

# Programs with Loops

## Loop invariants

- Each loop must be annotated with a loop invariant, @$L$
- `while` loop: $L$ must hold
  - at the beginning of each iteration before the loop condition is evaluated
- `for` loop: $L$ must hold
  - after the loop initialization, and
  - before the loop condition is evaluated

# Basic Paths: Loops

To handle loops, we break the function into basic paths.

## Basic Path

@ $\leftarrow$ precondition or loop invariant

finite sequence of instructions
(no loop invariants)

@ $\leftarrow$ loop invariant, assertion, or postcondition

# Basic Paths: Conditionals

## Basic paths split at conditionals

Replace each path $BP[\texttt{if } B \texttt{ then } S_1 \texttt{ else } S_2]$ by two paths

- $BP[\texttt{assume } B; S_1]$
- $BP[\texttt{assume } \neg B; S_2]$

## Semantics of "assume $B$"

Execution ends unless $B$ holds

# Example: LinearSearch

@pre $0 \le \ell \;\wedge\; u < a.length$
@post $rv \;\leftrightarrow\; \exists i.\, \ell \le i \le u \;\wedge\; a[i] = e$

```
bool LinearSearch(int[] a, int ℓ, int u, int e) {
  for
    @L : ℓ ≤ i ∧ (∀j. ℓ ≤ j < i → a[j] ≠ e)
    (int i := ℓ; i ≤ u; i := i + 1) {
    if (a[i] = e) return true;
  }
  return false;
}
```

# Example: Basic Paths of LinearSearch

**(1)**

@pre $0 \leq \ell \ \wedge \ u < a.length$

$i := \ell;$

@$L: \ \ell \leq i \ \wedge \ \forall j.\, \ell \leq j < i \ \rightarrow \ a[j] \neq e$

**(2)**

@$L: \ \ell \leq i \ \wedge \ \forall j.\, \ell \leq j < i \ \rightarrow \ a[j] \neq e$

`assume` $i \leq u;$

`assume` $a[i] = e;$

$rv$ := `true`;

@post $rv \ \leftrightarrow \ \exists j.\, \ell \leq j \leq u \ \wedge \ a[j] = e$

# Example: Basic Paths of LinearSearch

**(3)**

$@L : \ \ell \leq i \ \wedge \ \forall j.\, \ell \leq j < i \ \rightarrow \ a[j] \neq e$

$\texttt{assume}\ i \leq u;$

$\texttt{assume}\ a[i] \neq e;$

$i := i + 1;$

$@L : \ \ell \leq i \ \wedge \ \forall j.\, \ell \leq j < i \ \rightarrow \ a[j] \neq e$

**(4)**

$@L : \ \ell \leq i \ \wedge \ \forall j.\, \ell \leq j < i \ \rightarrow \ a[j] \neq e$

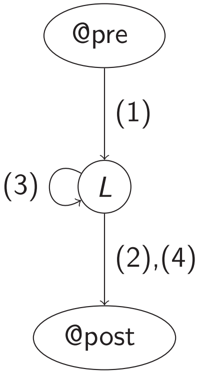
$\texttt{assume}\ i > u;$

$rv := \texttt{false};$

$@\text{post}\ rv \ \leftrightarrow \ \exists j.\, \ell \leq j \leq u \ \wedge \ a[j] = e$

# Example: Basic Paths of LinearSearch

Visualization of basic paths of LinearSearch



# Proving Partial Correctness

## Goal

- Prove that annotated function $f$ agrees with annotations
- Transform $f$ to finite set of verification conditions VC
- Validity of VC implies that function behaviour agrees with annotations

## Weakest precondition $\mathrm{wp}(F, S)$

- Informally: What must hold before executing statement $S$ to ensure that formula $F$ holds afterwards?
- $\mathrm{wp}(F, S)$ = weakest formula such that executing $S$ results in formula that satisfies $F$
- For all states $\sigma$ such that $\sigma \in \mathrm{wp}(F, S)$: successor state $\mathcal{S}[\![S]\!]\sigma \in F$.

# Proving Partial Correctness

## Weakest preconditions for each statement

- Assumption: What must hold before statement `assume` $B$ is executed to ensure that $F$ holds afterward?

$$\mathrm{wp}(F,\ \texttt{assume}\ B)\ \Leftrightarrow\ B\ \rightarrow\ F$$

- Assignment: What must hold before statement $x := e$ is executed to ensure that $F[x]$ holds afterward?

$$\mathrm{wp}(F[x],\ x := e)\ \Leftrightarrow\ F[e]$$

  ("substitute $x$ with $e$")

- Sequence of statements $S_1; \dots; S_n$ ($n > 1$),
  $\mathrm{wp}(F,\ S_1; \dots; S_n)\ \Leftrightarrow\ \mathrm{wp}(\mathrm{wp}(F,\ S_n),\ S_1; \dots; S_{n-1})$

# Proving Partial Correctness

## Verification condition of basic path

$$
\begin{array}{l}
@\ F \\
S_1; \\
\ldots \\
S_n; \\
@\ G
\end{array}
$$

is defined as

$$F \ \rightarrow \ \mathrm{wp}(G,\ S_1; \ldots; S_n)$$

This verification condition is often denoted by the Hoare triple

$$\{F\}S_1; \ldots; S_n\{G\}$$

# Proving Partial Correctness

## Approach

- Input: Annotated program
- Compute the set $P$ of all basic paths (finite)
- For all $p \in P$: generate verification condition $VC(p)$
- Check validity of $\bigwedge_{p \in P} VC(p)$

## Theorem

If $\bigwedge_{p \in P} VC(p)$ is valid, then each function agrees with its annotation.

# Example 1: VC of basic path

**(1)**

$@\ F:\ x \geq 0$
$S_1:\ x := x + 1;$
$@\ G:\ x \geq 1$

---

The VC is

$$F \;\rightarrow\; \text{wp}(G,\ S_1)$$

That is,

$$\begin{aligned} &\text{wp}(G,\ S_1) \\ &\Leftrightarrow\ \text{wp}(x \geq 1,\ x := x + 1) \\ &\Leftrightarrow\ (x \geq 1)\{x \mapsto x + 1\} \\ &\Leftrightarrow\ x + 1 \geq 1 \\ &\Leftrightarrow\ x \geq 0 \end{aligned}$$

Therefore the VC of path (1)

$$x \geq 0 \;\rightarrow\; x \geq 0\ ,$$

which is valid.

# Example 2: VC of basic path (2) of LinearSearch



**(2)**

$@L: \; F: \; \ell \leq i \;\wedge\; \forall j.\, \ell \leq j < i \;\rightarrow\; a[j] \neq e$
$S_1: \;$ `assume` $i \leq u;$
$S_2: \;$ `assume` $a[i] = e;$
$S_3: \; rv :=$ `true`;
$@\text{post } G: \; rv \;\leftrightarrow\; \exists j.\, \ell \leq j \leq u \;\wedge\; a[j] = e$

The VC is: $F \;\rightarrow\; \text{wp}(G,\; S_1; S_2; S_3)$

$$\begin{array}{l}
\text{wp}(G,\; S_1; S_2; S_3) \\
\Leftrightarrow\; \text{wp}(\text{wp}(rv \;\leftrightarrow\; \exists j.\, \ell \leq j \leq u \;\wedge\; a[j] = e,\; rv := \texttt{true}),\; S_1; S_2) \\
\Leftrightarrow\; \text{wp}(\texttt{true} \;\leftrightarrow\; \exists j.\, \ell \leq j \leq u \;\wedge\; a[j] = e,\; S_1; S_2) \\
\Leftrightarrow\; \text{wp}(\exists j.\, \ell \leq j \leq u \;\wedge\; a[j] = e,\; S_1; S_2) \\
\Leftrightarrow\; \text{wp}(\text{wp}(\exists j.\, \ell \leq j \leq u \;\wedge\; a[j] = e,\; \texttt{assume } a[i] = e),\; S_1) \\
\Leftrightarrow\; \text{wp}(a[i] = e \;\rightarrow\; \exists j.\, \ell \leq j \leq u \;\wedge\; a[j] = e,\; S_1) \\
\Leftrightarrow\; \text{wp}(a[i] = e \;\rightarrow\; \exists j.\, \ell \leq j \leq u \;\wedge\; a[j] = e,\; \texttt{assume } i \leq u) \\
\Leftrightarrow\; i \leq u \;\rightarrow\; (a[i] = e \;\rightarrow\; \exists j.\, \ell \leq j \leq u \;\wedge\; a[j] = e)
\end{array}$$

# Outline

- Proving partial correctness
    - Programs with recursive function calls

# Basic Paths: Recursive Function Calls

- Loops produce unbounded number of paths
  loop invariants cut loops to produce
  finite number of basic paths
- Recursive calls produce unbounded number of paths
  function specifications cut function calls

## Function specification

- Add function summary for each function call
- Instantiate pre- and postcondition with parameters of recursive call

# Example: BinarySearch

The recursive function BinarySearch searches subarray of sorted array $a$ of integers for specified value $e$.

sorted: weakly increasing order, i.e.

$$\text{sorted}(a, \ell, u) \ \Leftrightarrow \ \forall i, j.\ \ell \leq i \leq j \leq u \ \rightarrow \ a[i] \leq a[j]$$

## Function specifications

- Function postcondition (@*post*)
  It returns true iff $a$ contains the value $e$ in the range $[\ell, u]$
- Function precondition (@*pre*)
  It behaves correctly only if $0 \leq \ell$ and $u < a.length$

# Example: BinarySearch

```
@pre 0 ≤ ℓ ∧ u < a.length ∧ sorted(a, ℓ, u)
@post rv ↔ ∃i. ℓ ≤ i ≤ u ∧ a[i] = e
bool BinarySearch(int[] a, int ℓ, int u, int e) {
  if (ℓ > u) return false;
  else {
    int m := (ℓ + u) div 2;
    if (a[m] = e) return true;
    else if (a[m] < e) return BinarySearch(a, m + 1, u, e);
    else return BinarySearch(a, ℓ, m − 1, e);
  }
}
```

## Example: Binary Search with Function Call Assertions

@pre $0 \le \ell \ \wedge\ u < a.length \ \wedge\ \text{sorted}(a, \ell, u)$
@post $rv \ \leftrightarrow\ \exists i.\ \ell \le i \le u \ \wedge\ a[i] = e$
bool BinarySearch(int[] $a$, int $\ell$, int $u$, int $e$) {
  if ($\ell > u$) return false;
  else {
    int $m := (\ell + u)$ div 2;
    if ($a[m] = e$) return true;
    else if ($a[m] < e$) {
      @pre $0 \le m + 1 \ \wedge\ u < a.length \ \wedge\ \text{sorted}(a, m + 1, u)$;
      bool $tmp :=$ BinarySearch($a, m + 1, u, e$);
      @post $tmp \ \leftrightarrow\ \exists i.\ m + 1 \le i \le u \ \wedge\ a[i] = e$; return $tmp$;
    } else {
      @pre $0 \le \ell \ \wedge\ m - 1 < a.length \ \wedge\ \text{sorted}(a, \ell, m - 1)$;
      bool $tmp :=$ BinarySearch($a, \ell, m - 1, e$);
      @post $tmp \ \leftrightarrow\ \exists i.\ \ell \le i \le m - 1 \ \wedge\ a[i] = e$;
      return $tmp$;
    }
  }
}

# Summary

## Automatic verification of sequential programs

- Goal: Proof of partial correctness
- Program specification
  - Pre- and postconditions
  - Loop invariants
- Tools
  - Basic paths
  - Weakest precondition
  - Verification conditions
  - Function summaries